

# Bank regulation under fire sale externalities\*

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## Abstract

This paper examines the optimal design of and interaction between capital and liquidity regulations in a model characterized by systemic externalities generated by asset fire sales. We show that when capital is regulated but liquidity is not, banks still hold liquid assets to protect against liquidity shocks. Liquidity is advantageous from a macro-prudential standpoint as well: Higher liquidity holdings lead to less severe decreases in asset prices during times of distress. Individual banks do not internalize the fire sale externality and respond to the introduction of capital requirements by decreasing their liquidity ratios. Predicting this reaction from banks, the regulator raises the minimum capital ratio requirement to inefficiently high levels, which corresponds to a reduction in socially profitable long-term investments. Our results indicate that the pre-Basel III regulatory framework, which focused mainly on capital adequacy requirements, was both inefficient and ineffective in addressing systemic instability caused by liquidity shocks.

**Keywords:** Bank capital regulation, liquidity regulation, fire sales, Basel III

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# 1 Introduction

The recent financial crisis led to a redesign of bank regulations, with an emphasis on the macro-prudential aspects of regulation. Prior to the crisis, capital adequacy requirements were the dominant tool of bank regulators around the world. The crisis, however, revealed that even well capitalized banks can experience a deterioration of their capital ratios due to illiquid positions. The fact that several financial institutions faced liquidity constraints simultaneously created an urgent need for regulators and central banks to intervene in financial markets to restore stability. Without the unprecedented liquidity and asset price supports of leading central banks during the crisis, those liquidity problems could have resulted in a dramatic collapse of the financial system. The experience brought liquidity and its regulation into the spot light. A third generation of bank regulation principles, popularly known as Basel III, strengthens the previous Basel capital adequacy accord by adding macro-prudential aspects and liquidity requirements such as the Liquidity Coverage Ratio (LCR) and Net Stable Funding Ratio (NSFR).

Several countries, including the US and the EU countries, have already adopted Basel III liquidity requirements together with the enhanced capital requirements. However, the guidance from theoretical literature on the regulation of liquidity and the interaction between liquidity and capital regulations is quite limited. The scarcity of academic guidance is apparent in a 2011 survey paper on illiquidity by Jean Tirole, which succinctly asks: “Can we trust the institutions to properly manage their liquidity, once excessive risk taking has been controlled by the capital requirement?” [Tirole \(2011\)](#).

In this paper we investigate the optimal design of capital and liquidity regulations, and the interaction between the two, in a model characterized by systemic externalities generated by asset fire sales. We consider a three-period model in which a continuum of banks have access to two types of assets at the initial period. The risky asset has a constant return but with a known probability it requires additional investment in the future before collecting returns. We call this additional liquidity need the liquidity shock. The liquid asset provides zero net return; however, it can be used to cover the cost of additional investment for risky assets. Thus, banks have to decide in the initial period how much risky and liquid assets to carry in their portfolio.

In case of a liquidity shock, banks can use their liquid asset holdings to cover additional costs. A collateral constraint prevents banks from raising additional external finance in the interim period. Therefore, if liquidity from the initial period is not enough to offset the shock, banks’ only other option is to sell some of the risky assets to outside investors to save the remaining risky assets.<sup>1</sup> Risky assets are sold at fire sales because outside investor demand for risky assets slopes downward. The outside investors are less productive, and unlike banks their marginal productivity decreases

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<sup>1</sup>The liquidity shock is aggregate in nature, therefore the liquidity need cannot be satisfied within the banking system as all the banks are in need of liquidity. This assumption is not crucial for the results. In Section 5 we study the case with idiosyncratic shocks.

as they purchase more assets. Thus, outside investors offer a lower price when banks try to sell more risky assets. A lower price requires each bank to sell more risky assets, creating an externality that goes through asset prices.

Atomistic banks do not take into account the effect of their initial portfolio choices on the fire sale price. If banks hold more risky asset, the liquidity need in case of an aggregate shock is greater. As a result there are more fire sales and a lower fire sale price, which in turn requires each bank to sell more risky asset to raise a given amount of liquid resources. Because of the fire sale externality that banks exert on each other, banks' initial portfolio choices are not socially optimal. In our framework, this is the only externality. We do not model agency or information problems that the literature has traditionally used to justify capital or other bank regulations. We show that under the fire sale externality alone, the first welfare theorem fails and there is a need for bank regulation. The regulation required is macro-prudential because it should address the instability in the banking system by targeting aggregate capital and liquidity ratios.

Although the probability and size of a liquidity shock are exogenous in the model, we show that whether fire sales take place in equilibrium and the amount of fire sales are endogenously determined. In principle, banks can perfectly insure themselves against fire sale risk by holding sufficiently high liquidity. However, we show that such insurance is never optimal. The intuition is straightforward. The marginal benefit of holding liquid assets exceeds unity as long as there are fire sales, and it decreases with the amount of liquidity. Perfect insurance guarantees that no fire sale takes place, and as a result the marginal return of liquid assets is equal to one which is dominated by the net marginal return on risky assets. In other words, there is no need to hoard any liquidity when there is no fire sale (risk). Therefore, there is an optimal interior level of liquidity ratio for which the private (or social in the case of constrained planner's problem) cost and benefit of liquidity are equalized.

Second, we investigate whether capital requirements would alone be sufficient to address the systemic externalities generated by fire sales, or if additional introduction of liquidity regulation could further improve financial stability and welfare. We show that banks respond to the introduction of capital regulations by decreasing their liquidity ratios. If there is no regulation, banks choose a composition of risky and safe assets in their portfolio that reflects their privately optimal level of risk taking. When the level of risky investment is limited by capital regulations, banks reduce the liquidity of their portfolio in order to get closer to their privately optimal level of fire sale risk. This is in a sense an unintended consequence of the capital regulation: Capital regulation improves financial stability by limiting risky investment, which in turn weakens banks' incentive to hold sufficient liquidity.

Next, we study regulation of both capital and liquidity ratios as in Basel III (complete regulation) and compare this framework to the capital regulation regime (partial regulation). We show that under partial regulation, bank capital ratios are inefficiently high. Regulators will tighten

capital regulations under a capital ratio regime to offset the banks' lower liquidity ratios, reducing socially profitable long-term investments. We also show that bank liquidity under competitive equilibrium and partial regulation is inefficiently low. Banks hold liquid assets for micro-prudential reasons even if there is no regulation on liquidity because they can use these resources to protect against liquidity shocks. Liquidity is advantageous from a macro-prudential standpoint as well: Higher liquidity holdings lead to less severe decreases in asset prices during times of distress. However, banks fail to internalize this macro-prudential aspect of liquidity.

The lack of complementary liquidity ratio requirements leads to inefficiently low levels of long-term investments and more severe financial crises, undermining the purpose of capital adequacy requirements. Our results indicate that the pre-Basel III regulatory framework, with its focus on capital adequacy requirements, was inefficient and ineffective in addressing systemic instability caused by liquidity shocks. Therefore, our results indicate that Basel III liquidity regulations are a step in the right direction.

Our contribution is three folds. First, to the best of our knowledge, this is one of the first papers to study the interaction between capital and liquidity regulations. Second, we contribute to the fire sales literature by introducing an explicit role for safe assets and showing that even though banks can perfectly hedge against fire sale risk by holding sufficient liquidity, they still choose to take some of this risk. Third, the paper contributes to the theory of economic policy making. We show that even though there is only a fire sales externality, a single regulatory tool is not enough to achieve the socially optimal level of fire sales. This result complements the Tinbergen rule which argues that the number of policy tools must be at least as much as the number of policy objectives. The target of the regulator in this model is to maximize the expected welfare. Our results show that the regulator cannot reach this target by using capital regulations alone because banks have two channels to access fire sale risk, the capital and liquidity channels. We show that if capital is regulated but liquidity is not, banks react by decreasing their liquidity ratios. Therefore, establishing the social optimum always requires using both capital and liquidity regulatory tools.

The paper proceeds as follows. Section 2 contains a brief summary of related literature. Section 3 provides the basics of the model and presents the solutions for the equilibrium of both unregulated and regulated economies. Section 4 compares and contrasts the equilibrium outcomes three cases: competitive equilibrium (no regulation), partially regulated equilibrium (only capital regulation), and the complete regulation equilibrium (both capital and liquidity regulations). Section 5 investigates the robustness of the results to some changes in the model environment. Section 6 concludes. The appendix contains the closed-form solutions of the model and proofs omitted in the main text.

## 2 Literature review

Even though capital and liquidity regulations have been studied extensively on their own, we are aware of only a few papers that investigate the interaction between these two classical tools of

regulators and their optimal determination. [Kashyap, Tsomocos, and Vardoulakis \(2014\)](#) consider an extended version of the [Diamond and Dybvig \(1983\)](#) model to investigate the effectiveness of several bank regulations in addressing two common financial system externalities:<sup>2</sup> excessive risk-taking due to limited liability and bank-runs. The central message of the paper is that a single regulation alone is never sufficient to correct for the inefficiencies created by these two externalities. Unlike this paper, their paper does not consider fire sale externalities. This causes a divergence in our results as well. For example, in their paper optimal regulation does not necessarily involve capital or liquidity regulations.

[Walther \(2014\)](#) also studies macro-prudential regulation in a model characterized pecuniary externalities due to fire sales. In his setup the fire sale price is exogenously fixed and the socially optimal outcome is to have “no fire sales” in equilibrium unlike our paper where partial fire sales are allowed and even the socially optimal solution requires taking some fire sale risk. [Walther \(2014\)](#) shows that both macro-prudential regulation and Pigouvian taxation can achieve the “no fire sales” outcome; however, implementation of Pigouvian taxation requires more information. Pigouvian taxation serves as an important theoretical benchmark, yet it is not part of the toolkit designed by the Basel Committee, which is the focus of this paper.

[De Nicoló, Gamba, and Lucchetta \(2012\)](#) consider a dynamic model of bank regulation and show that liquidity requirements, when added to capital requirements, eliminate the benefits of mild capital requirements by hampering bank maturity transformation, and hence result in lower bank lending, efficiency, and social welfare. In their model, liquidity is only welfare reducing because, unlike our paper, they do not consider the role of liquidity in correcting negative externalities arising from fire sales.

Even though the literature on the interaction between capital and liquidity requirements is limited, there are studies that examine the interaction between different tools available to regulators. [Acharya, Mehran, and Thakor \(2010\)](#) show that the optimal capital regulation requires a two-tiered capital requirement with a part of bank capital invested in safe assets. The special capital should be unavailable to creditors upon failure so as to retain market discipline and be available to shareholders only contingent on good performance in order to contain risk-taking.

[Acharya \(2003\)](#) shows that convergence in international capital adequacy standards cannot be effective unless it is accompanied by convergence in other aspects of banking regulation, such as closure policies. Externalities in his model are of the form of the cost of investment in a risky asset. He assumes that a bank in one country increases costs of investment for itself and for a bank in the other country as it invests more in the risky asset and thereby creates externalities for the bank in the neighboring country.

[Hellmann, Murdock, and Stiglitz \(2000\)](#) show that while capital requirements can induce prudent behavior, they lead to Pareto-inefficient outcomes by reducing banks’ franchise values, and

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<sup>2</sup>They consider following regulations: deposit insurance, loan-to-value limits, dividend taxes, capital and liquidity ratio requirements.

hence providing incentives for gambling. Pareto-efficient outcomes can be achieved by adding deposit-rate controls as a regulatory instrument. The latter restores the prudent behavior by increasing franchise values. Similar to their result we show that capital requirements provide Pareto-efficiency only if they are combined with the liquidity requirements.

A number of papers, especially after the global financial crisis, drew attention to the macro-prudential role of liquidity requirements, similar to that considered in this paper. [Calomiris, Heider, and Hoerova \(2013\)](#) argue that the role of liquidity requirements should be conceived not only as an insurance policy that addresses the liquidity risks in distressed times as proposed by Basel III, but also as a prudential regulatory tool to make crises less likely. However, their paper does not analyze how the liquidity requirements interact with prudential capital regulations.

This paper is also related to the literature that features financial amplification and asset fire sales which includes the seminal contributions of [Fisher \(1933\)](#), [Bernanke and Gertler \(1989\)](#), [Kiyotaki and Moore \(1997\)](#), [Krishnamurthy \(2003, 2010\)](#), and [Brunnermeier and Pedersen \(2009\)](#). In our model, fire sales result from the combined effects of asset-specificity and correlated shocks that hit an entire industry or economy. This idea, originating with [Williamson \(1988\)](#) and [Shleifer and Vishny \(1992\)](#), is employed by fire sale models such as [Lorenzoni \(2008\)](#), [Korinek \(2011\)](#), [Stein \(2012\)](#), and [Kara \(2015\)](#). These papers show that under pecuniary externalities arising from asset fire sales, there exists over-borrowing and hence over-investment in risky assets in a competitive setting compared to the socially optimal solution. However, unlike our paper, none of these papers gives an explicit role for safe assets, which banks can use to completely insure themselves against the fire sale risk.

The constrained inefficiency of competitive markets in this paper is due to the existence of pecuniary externalities under incomplete markets. The Pareto suboptimality of competitive markets when the markets are incomplete goes back at least to the work of [Borch \(1962\)](#). The idea was further developed in the seminal papers of [Hart \(1975\)](#), [Stiglitz \(1982\)](#), and [Geanakoplos and Polemarchakis \(1986\)](#), among others. [Greenwald and Stiglitz \(1986\)](#) extended the analysis by showing that, in general, pecuniary externalities by themselves are not a source of inefficiency, but can lead to significant welfare losses when markets are incomplete or there is imperfect information.

In this paper, the incompleteness of markets arises from the financial constraints of bankers in the interim period. In particular, similar to [Kiyotaki and Moore \(1997\)](#) and [Korinek \(2011\)](#), we assume that a commitment problem prevents banks from borrowing the funds necessary for restructuring when liquidity shocks hit. If the markets are complete and banks can borrow by pledging the future return stream from the assets, fire sales are avoided. In this first best world, there is no need for either capital or liquidity requirements because a systemic externality in the financial markets no longer exists.

### 3 Model

The model consists of three periods,  $t = 0, 1, 2$ ; along with a continuum of banks and a continuum of consumers each with a unit mass and a financial regulator. There is also a unit mass of outside investors. All agents are risk-neutral and derive utility from consumption in the initial and final periods.

There are two types of goods in this economy, a consumption good and an investment good (that is, the liquid and the illiquid asset). Consumers are endowed with  $e$  units of consumption goods at  $t = 0$  but none at  $t = 1$  and  $t = 2$ .<sup>3</sup> Each bank is endowed with  $E$  units equity capital at  $t = 0$  in terms of consumption goods. Banks have a technology that converts consumption goods into investment goods one-to-one at  $t = 0$ . Investment goods that are managed by a bank until the last period yield  $R > 1$  consumption goods per unit.<sup>4</sup> The investment good can never be converted back into the consumption good and it fully depreciates after the return is collected at  $t = 2$ .

Banks choose at  $t = 0$ , how much risky assets to hold, denoted by  $n_i$ , and how much liquid (safe) assets to put aside, denoted by  $b_i$ , for each unit of risky assets. The total amount of liquid asset hold by each bank is then  $n_i b_i$ , and  $b_i$  can be interpreted as a liquidity ratio. The return on the liquid asset is normalized to one. Therefore, the total asset size of a bank is  $n_i + n_i b_i = (1 + b_i)n_i$ . On the liability side, each bank raises  $L_i = (1 + b_i)n_i - E$  units of consumption goods from consumers at  $t = 0$  to finance these assets.

We assume that the initial equity of banks is sufficiently large to avoid default in the bad state equilibrium. As a result, the deposit are safe, and hence consumers inelastically supply deposits to banks at net zero interest rate at the initial period. This assumption also allows us to focus only one friction, that is, fire sale externalities, and study the implications of this friction for optimal regulation of bank capital and liquidity. However, as we show in Section 5, our results are robust to relaxing this assumption and allowing bank default in equilibrium.

We assume that there is a non-pecuniary cost of operating a bank, captured by  $\Phi((1+b_i)n_i)$ . The non-pecuniary operational cost of a bank is increasing in the size of the balance sheet,  $\Phi'(\cdot) > 0$ , and it is convex,  $\Phi''(\cdot) > 0$ . This assumption, similar to the ones imposed by [Van den Heuvel \(2008\)](#) and [Acharya \(2003, 2009\)](#), ensures that banks' problem is well defined and there is an interior solution to this problem. The convex operational cost assumption allows us to have banks with flexible balance sheet size in the model. If the balance sheet size of the bank is fixed, and liquid and risky (illiquid) assets are the only assets a bank can buy, the choice between the liquid and the illiquid (risky) boils down to a single choice, namely an allocation problem. If a bank decreases its risky assets, the amount of liquid asset in the bank's portfolio increases because now there are

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<sup>3</sup>We assume that the initial endowment of consumers is sufficiently large, and it is not a binding constraint in equilibrium.

<sup>4</sup>To simplify the exposition, we abstract from modeling the relationship between banks and firms. Instead we assume that banks directly invest in physical projects. This is equivalent to assuming that there are no contracting frictions between banks and firms as more broadly discussed in [Stein \(2012\)](#).

more resources available for the liquid asset. In our framework with flexible balance sheets, banks can increase or decrease the amount of risky and liquid assets simultaneously, if it is optimal for them to do so. As a result, banks have two independent choices and this setup allows us to study two separate regulations, namely capital and liquidity regulations, that address these two choices.

Investment and deposit collection decisions are made at time  $t = 0$ . The only uncertainty in the model is regarding the risky asset and it is resolved at the beginning of  $t = 1$ : The economy lands in good times with probability  $1 - q$ , and in bad times with probability  $q$ . In good times no banks are hit with shocks, therefore no further actions are taken. Banks keep managing their investment goods and realize a total return of  $Rn_i + n_i b_i$  to their portfolio of risky and safe assets in the last period. However, in bad times, the risky asset of all banks are distressed. In case of distress, the investment (risky assets) has to be restructured in order to remain productive. Restructuring costs are equal to  $c \leq 1$  units of consumption goods per unit of the risky asset. If  $c$  is not paid, the risky investment is scrapped (that is, it fully depreciates).

A bank can use the liquid assets hoarded from the initial period,  $n_i b_i$ , to carry out the restructuring of distressed investment at  $t = 1$ . However, if the liquid assets are not sufficient to cover the entire cost of restructuring, the bank will need external finance. Other than banks, only outside investors are endowed with liquid resources at this point. Due to a commitment problem, banks cannot borrow the required resources from outside investors. In particular, similar to [Kiyotaki and Moore \(1997\)](#) and [Korinek \(2011\)](#), we assume that banks can only pledge the market value but not the dividend income of their asset holdings next period to outside investors. This assumption prevents banks from borrowing between the interim and the last periods because the value of all assets are zero in the final period, and hence banks have no collateral to pledge to outside investors in the interim period.<sup>5</sup> In other words, this assumption states that the contracts between banks and outside investors are not enforceable. The only way for banks to raise necessary funds for restructuring is to sell some fraction of the risky asset to outside investors in an exchange of consumption goods.<sup>6</sup>

The asset sales by banks are in the form of fire sales: The risky asset is traded below its fundamental value for banks, and the price decreases as banks try to sell more assets. Banks retain

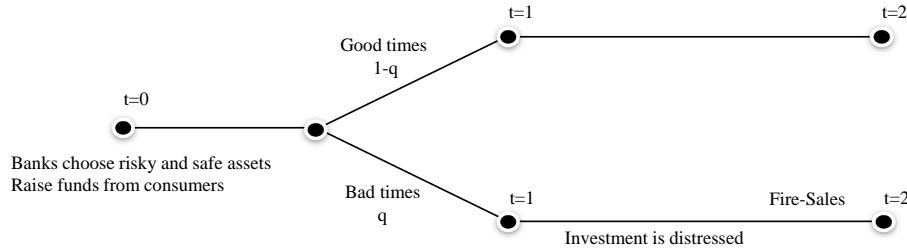
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<sup>5</sup>For simplicity, we assume that the commitment problem is extreme (that is, banks cannot commit to pay any fraction of their production to outside investors). Assuming a milder but sufficiently strong commitment problem where banks can commit a small fraction of their production, as in [Lorenzoni \(2008\)](#) and [Gai et al. \(2008\)](#), does not change the qualitative results of this paper. On the other hand, if we complete the markets by allowing banks to borrow from outside investors by pledging the future return stream from the assets, there would not be a reason for fire sales and the first best world would be established. In the first best world there would not be a need for regulation the pecuniary externality in financial markets would be eliminated.

<sup>6</sup>An alternative story would be that households come in two generations as in [Korinek \(2011\)](#) and the assets produce a (potentially risky) return in the interim period in addition to the safe return in the final period. In this case, banks can borrow from the first generation households at the initial period because they have sufficient collateral to back their promises in the interim period, but banks cannot borrow from second generation households because the value of all assets are zero in the final period. In this alternative story, second generation households will be the buyers of assets from banks, and they will employ them in a less productive technology to produce returns in the final period similar to outside investors here.



Figure 1: Timing of the model



only a fraction  $\gamma$  of their assets after fire sales, which depends on banks' liquidity shortage as well as the fire sale price of assets. The sequence of events is illustrated in Figure 1.

We first solve the competitive equilibrium of the model when there is no regulation on banks. Second, we consider a partially regulated economy in which there is only capital regulation but there is no regulation on bank liquidity ratios. Last, we consider the complete regulation, in which case the regulator imposes both a capital and liquidity requirement on banks. In particular, the liquidity regulation requires the banks to satisfy a minimum liquidity ratio such that  $b_i \geq \underline{b}$ . The capital regulation requires bank to satisfy a minimum risk-weighted capital ratio,  $\underline{k}$ , at  $t = 0$ , such that  $k_i = E/n_i \geq \underline{k}$ . Because the inside equity of banks,  $E$ , is fixed in our model, the minimum risk-weighted capital ratio regulation is equivalent to a regulation in the form of an upper limit on initial risky investment levels,  $\bar{n}$ , such that banks' investments have to satisfy  $n_i \leq \bar{n}$ , where  $\bar{n} \equiv E/\underline{k}$ . For analytical convenience we use the upper bound on risky investment formulation for capital regulation in the rest of the paper.

### 3.1 Crisis and fire sales

The decision of agents at time  $t = 0$  depend on the expectations regarding the events at time  $t = 1$ . Thus applying the solution by backwards induction, we first analyze the equilibrium at the interim period in each state of the world for a given set of investment levels. We then study the equilibrium at  $t = 0$ . Note that if the good state is realized at  $t = 1$ , banks take no further action and obtain a total return of  $\pi_i^{Good} = Rn_i + b_in_i$  at the final period,  $t = 2$ . Therefore, for the interim period  $t = 1$ , studying the equilibrium only for bad times is sufficient. We start with the problem of outside investors in bad times, and then analyze the problem of banks.

#### 3.1.1 Outside investors

The outside investors are endowed with large resources of consumption goods at  $t = 1$  and they can purchase investment goods from the banks. Let us denote the amount of investment goods they buy from the banks by  $y$ . Having a concave production technology, the outside investors employ these investment goods to produce  $F(y)$  units of consumption goods at  $t = 2$ . Let  $P$  denote the

market price of the investment good in bad times at  $t = 1$ .<sup>7</sup> Each outside investor takes the market price as given and chooses how much investment goods to buy,  $y$ , in order to maximize net returns from investment at  $t = 2$ :

$$\max_{y \geq 0} F(y) - Py. \quad (1)$$

The first order condition of the investors maximization problem,  $F'(y) = P$ , determines the outside investors' (inverse) demand function for the investment good. Using this, we can define their demand function  $Q^d(P)$  as follows:  $Q^d(P) \equiv F'(P)^{-1} = y$ .

**Assumption 1** (*Concavity*).  $F'(y) > 0$  and  $F''(y) < 0$  for all  $y \geq 0$ , with  $F'(0) \leq R$ .

The *Concavity* assumption establishes that outside investors are less efficient than the banks. Outside investors' return is strictly increasing the amount of assets employed, ( $F'(y) > 0$ ), and they face decreasing returns to scale in the production of consumption goods, ( $F''(y) < 0$ ), as opposed to banks that are endowed with a constant returns to scale technology as described above. Together with concavity,  $F'(0) \leq R$  implies that outside investors are less productive than banks at each level of investment goods employed. The concavity of the return function implies that the demand function of outside investors for investment goods is downward sloping (see Figure 2). In other words, outside investors will require higher discounts to absorb more assets from distressed banks at  $t = 1$ . The decreasing returns to scale technology assumption is a reduced way of modeling the existence of industry-specific heterogeneous assets, similar to [Kiyotaki and Moore \(1997\)](#), [Lorenzoni \(2008\)](#), [Korinek \(2011\)](#), and [Stein \(2012\)](#). In this more general setup, outside investors would first purchase assets that are easy to manage, but as they purchase more assets, they would need to buy the ones which require more and more sophisticated management and operation skills.

The idea that some assets are industry-specific, and hence less productive in the hands of outsiders, has its origins in [Williamson \(1988\)](#) and [Shleifer and Vishny \(1992\)](#).<sup>8</sup> These studies claim that when major players in such industries face correlated liquidity shocks and cannot raise external finance due to debt overhang, agency, or commitment problems, they may have to sell assets to outsiders. Outsiders are willing to pay less than the value in best use for the assets of distressed enterprises because they do not have the specific expertise to manage these assets well and therefore face agency costs of hiring specialists to run these assets.<sup>9</sup>

Empirical and anecdotal evidence suggests the existence of fire sales of physical as well as financial assets. Using a large sample of commercial aircraft transactions, [Pulvino \(2002\)](#) shows that distressed airlines sell aircraft at a 14 percent discount from the average market price. This

<sup>7</sup>Price of the investment good at  $t = 0$  will be one as long as there is positive investment, and the price at  $t = 2$  will be zero because the investment good fully depreciates at this point.

<sup>8</sup>Industry-specific assets can be physical or they can be portfolios of financial intermediaries ([Gai et al., 2008](#)).

<sup>9</sup>As opposed to the asset specificity idea discussed above, in [Allen and Gale \(1994, 1998\)](#) and [Acharya and Yorulmazer \(2008\)](#) the reason for fire sales is the limited amount of available cash in the market to buy long-term assets offered for sale by agents who need liquid resources immediately. The scarcity of liquid resources leads to necessary discounts in asset prices, a phenomenon known as "cash-in-the-market pricing."

discount exists when the airline industry is depressed but not when it is booming. [Coval and Stafford \(2007\)](#) show that fire sales exist in equity markets when mutual funds engage in sales of similar stocks.

We need to impose some more structure on the return function of outside investors in order to ensure that the equilibrium of this model exists and is unique.

**Assumption 2** (*Elasticity*).

$$\epsilon^d = \frac{\partial Q^d(P)}{\partial P} \frac{P}{Q^d(P)} = \frac{F'(y)}{yF''(y)} < -1 \quad \text{for all } y \geq 0$$

The *Elasticity* assumption says that outside investors' demand for the investment good is elastic. This assumption implies that the amount spent by outside investors on asset purchases,  $Py = F'(y)y$ , is strictly increasing in  $y$ . Therefore we can also write the *Elasticity* assumption as  $F'(y) + yF''(y) > 0$ . If this assumption was violated, multiple levels of asset sales would raise a given amount of liquidity, and multiple equilibria in the asset market at  $t = 1$  would be possible. This assumption is imposed by [Lorenzoni \(2008\)](#) and [Korinek \(2011\)](#) in order to rule out multiple equilibria under fire sales.<sup>10</sup>

**Assumption 3** (*Regularity*).  $F'(y)F'''(y) - 2F''(y)^2 \leq 0$  for all  $y \geq 0$ .

The *Regularity* assumption holds whenever the demand function of global investors is log-concave, but it is weaker than log-concavity.<sup>11</sup> Log-concavity of a demand function is a common assumption used in the Cournot games literature<sup>12</sup>. This assumption ensures the existence and uniqueness of an equilibrium in a simple  $n$ -player Cournot game. The assumption appears in [Kara \(2015\)](#) as well, and following him we call it a “regularity” assumption on  $F$ . In our setup this assumption ensures that the objective functions of regulators are well-behaved. It will be crucial proving some key results of our paper.<sup>13</sup>

**Assumption 4** (*Technology*).  $1 + qc < R < 1/(1 - q)$ .

<sup>10</sup>[Gai et al. \(2008\)](#) provide the leading example where this assumption is not imposed and multiple equilibria in the asset market are therefore considered. The authors assume that which equilibrium is chosen is determined by the ex-ante beliefs of agents. They show that under both pessimistic and optimistic beliefs, the competitive equilibrium is constrained inefficient and exhibits over-borrowing and over-investment compared to the planner's solution.

<sup>11</sup>A function is said to be log-concave if the logarithm of the function is concave. Let  $\phi(y) \equiv F'(y)$  denote the (inverse) demand function of global investors. We can rewrite this assumption as  $\phi(y)\phi''(y) - 2\phi'(y)^2 \leq 0$ . We can show that the demand function is log-concave if and only if  $\phi(y)\phi''(y) - \phi'(y)^2 \leq 0$ . Clearly the *Regularity* assumption holds whenever the demand function is log-concave. However, it is weaker than log-concavity and may also hold if the demand function is log-convex (that is, if  $\phi(y)\phi''(y) - \phi'(y)^2 \geq 0$ ).

<sup>12</sup>Please see [Amir \(1996\)](#).

<sup>13</sup>Many regular return functions satisfy conditions given by the *Concavity*, *Elasticity* and *Regularity* assumptions. Here are two examples that satisfy all three of the above assumptions:  $F(y) = R \ln(1+y)$  and  $F(y) = \sqrt{y + (1/2R)^2}$ . The following example satisfies the *Concavity* assumption, but not the *Elasticity* and *Regularity* assumptions:  $F(y) = y(R - 2\alpha y)$  where  $2\alpha y < R$  for all  $y \geq 0$ .

The first inequality in the *Technology* assumption states that the net expected return on the risky asset is positive. As described,  $R$  stands for the  $t = 2$  return on the risky asset which requires one unit investment in terms of consumption goods at  $t = 0$ . The expected cost of restructuring is equal to  $qc$ , where  $q$  is the restructuring cost which arrives with a probability  $q$ . The second inequality,  $R < 1/(1 - q)$  ensures that the price of assets in the bad state,  $P$ , is greater than the liquidity shock,  $c$ , in equilibrium.

### 3.1.2 Banks' problem in the bad state

Consider the problem of bank  $i$  when bad times are realized at  $t = 1$ . The bank has an investment level  $n_i$  and liquid assets of  $b_i n_i$  chosen at the initial period. If  $b_i \geq c$ , the bank has enough liquid resources to restructure all of the assets. In this case, the bank obtains a gross return of  $Rn_i + (b_i - c)n$  on its portfolio at  $t = 2$ . However, if  $b_i < c$ , then the bank does not have enough liquid resources to cover the restructuring costs entirely. In this case, the bank decides what fraction of these assets to sell  $(1 - \gamma_i)$  to generate the additional resources for restructuring.<sup>14</sup> Note that  $\gamma_i$  then represent the fraction of assets that a bank keeps after fire sales.<sup>15</sup> Thus, the bank takes the price of the investment good ( $P$ ) as given and chooses  $\gamma_i$  to maximize total returns from that point on

$$\pi_i^{Bad} = \max_{0 \leq \gamma_i \leq 1} R\gamma_i n_i + P(1 - \gamma_i)n_i + b_i n_i - cn_i, \quad (2)$$

subject to the budget constraint

$$P(1 - \gamma_i)n_i + b_i n_i - cn_i \geq 0. \quad (3)$$

The first term in (2) is the total return to be obtained from the unsold part of the assets. The second term is the revenue raised by selling a fraction  $(1 - \gamma_i)$  of the assets at the given market price  $P$ . The third term is the liquid assets hoarded at  $t = 0$ . The last term,  $cn_i$ , gives the total cost of restructuring. Budget constraint (3) says that the sum of the liquid assets carried from the initial period and the revenues raised by selling assets must be greater than or equal to the restructuring costs.

By the *Concavity* assumption, the equilibrium price of assets must satisfy  $P \leq F'(0) \leq R$ , otherwise outside investors will not purchase any assets. In equilibrium, we must also have  $P > c$ , otherwise in the bad state banks would not sell assets, that is there would not be any asset

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<sup>14</sup>For example, if the assets are physical, restructuring costs can be maintenance costs or working-capital needs. For financial assets,  $cn_i$  can be considered as a liquidity shock generated by margin calls which could result from changing risk perceptions or appetite.

<sup>15</sup>Following Lorenzoni (2008) and Gai et al. (2008), we assume that banks have to restructure an asset before selling it. Basically, this means that bank receive the asset price  $P$  from outside investors, use a part,  $c$ , to restructure the asset, and then deliver the restructured assets to the investors. Therefore banks will sell assets only if  $P$  is greater than the restructuring cost,  $c$ . We could assume, without qualitatively changing our results, that it is the responsibility of outside investors to restructure the assets that they purchase.

supply for outsiders to buy. However, if there is no supply there is an incentive for each bank to deviate and to sell some assets to outsiders. The deviating bank will receive a price close to  $F'(0) \leq R$ , which is better than discarding the assets because similar to [Lorenzoni \(2008\)](#) we also assume that  $F'(0) > c$ . Having  $P > c$  also implies that scrapping of investment goods never arises in equilibrium.

The choice variable,  $\gamma_i$ , affects only the first two terms in the expected return function of banks in (2), whereas the last terms are predetermined in the bad state at  $t = 1$ . The continuation return is, therefore, actually a weighted average of  $R$  and  $P$  where weights are  $\gamma_i$  and  $1 - \gamma_i$ , respectively. Banks want to choose  $\gamma_i$  as high as possible because by keeping assets on the balance sheet they get  $R$  whereas by selling them they get  $P \leq R$ . Therefore, banks sell assets just enough to cover their liquidity shortage,  $cn_i - b_in_i$ . This means that the budget constraint binds, from which we can obtain  $\gamma_i = 1 - (c - b_i)/P \in (0, 1)$ . As a result, the fraction of investment goods sold by each bank is:

$$1 - \gamma_i = \frac{c - b_i}{P} \in (0, 1). \quad (4)$$

The fraction of assets sold ( $1 - \gamma_i$ ) is decreasing in the price of the investment good, ( $P$ ), and in liquidity ratio,  $b_i$ , and increasing in the cost of restructuring ( $c$ ). Therefore, the supply of investment goods by each bank  $i$  is equal to:

$$Q_i^s(P, n_i, b_i) = (1 - \gamma_i)n_i = \frac{c - b_i}{P}n_i, \quad (5)$$

for  $c < P \leq R$ . This supply curve is downward-sloping and convex, which is standard in the fire sales literature (see, [Figure 2](#), left panel). A negative slope implies that if there is a decrease in the price of assets, banks have to sell more assets in order to generate the certain amount of resources needed for restructuring. Liquidity ratio of a bank,  $b_i$ , also negatively affects its asset supply in the bad state, as can be seen in (5), because a higher liquidity ratio allows a bank to offset a larger fraction of the shock using banks' own resources.

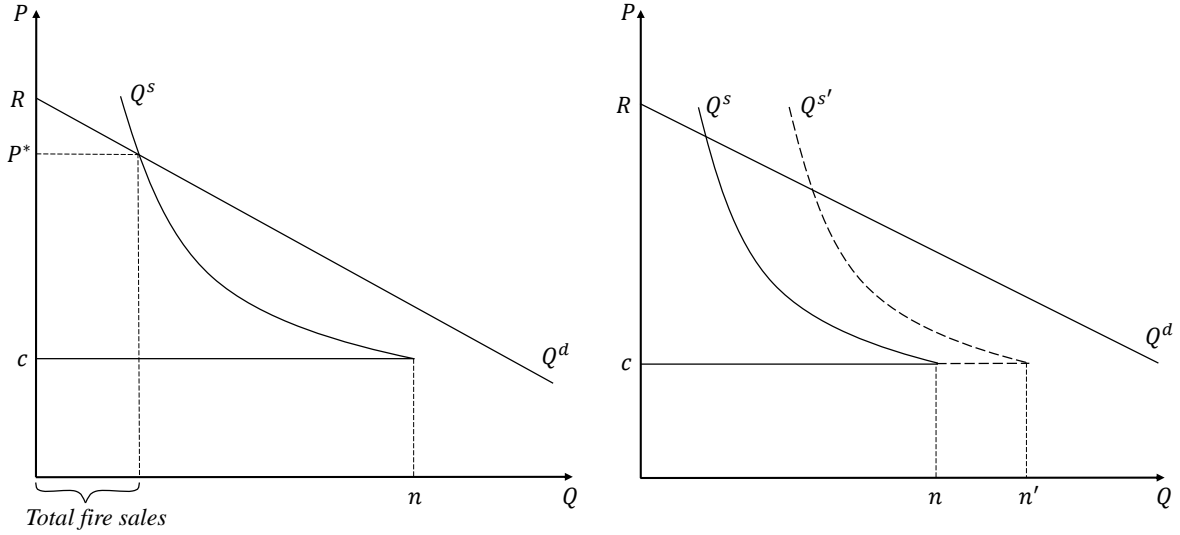
We can substitute the optimal value of  $\gamma_i$  using (4) into (2) and write the maximized total returns of banks in the bad state at  $t = 1$  as  $\pi_i^{Bad} = R\gamma_in_i = R(1 - \frac{c-b_i}{P})n_i$  for a given  $n_i$ . Note that the sum of the last three terms in (2) is zero at the optimal choice of  $\gamma_i$  because of the binding budget constraint.

### 3.1.3 Asset market equilibrium at date 1

Equilibrium price of investment goods in the bad state,  $P$ , will be determined by the market clearing condition

$$E(P, n, b) = Q^d(P) - Q^s(P, n, b) = 0 \quad (6)$$

Figure 2: Equilibrium in the investment goods market and comparative statics



This condition says that the excess demand in the asset market, denoted by  $E(P, n, b)$ , is equal to zero at the equilibrium price.  $Q^d(P)$  is the demand function that was obtained from the first order conditions of the outside investors' problem, given by (1).  $Q^s(P, n, b)$  is the total supply of investment goods obtained by aggregating the asset supply of each bank given by (5). This equilibrium is illustrated in the left panel of Figure 2. Note that the equilibrium price of the investment good, and the amount of fire sales, at  $t = 1$  are functions of the initial total investment in the risky asset and the aggregate liquidity ratio. Therefore, from the perspective of the initial period, we denote the equilibrium price as  $P(n, b)$ . How does a change in the initial investment level or the liquidity ratio affect the price of the investment good at  $t = 1$  and the extent of fire sales? Lemma 1 addresses these questions.

**Lemma 1.** *The fire sale price of assets at  $t = 1$ ,  $P(n, b)$ , is decreasing in  $n$  and increasing in  $b$ .*

Lemma 1 implies that higher investment in the risky asset or a lower liquidity ratio increases the severity of the financial crisis by lowering the asset prices. This effect is illustrated in the right panel of Figure 2. Suppose that the banks enter the interim period with greater holdings of risky asset. In this case, banks have to sell more assets at each price, as shown by the supply function given by (5), because the total cost of restructuring,  $cn_i$ , is increasing in the amount of initial risky assets,  $n_i$ . Graphically, the aggregate supply curve shifts to the right, as shown by the dotted-line supply curve in the right panel of Figure 2, which causes a decrease in the equilibrium price of investment goods. A lower initial liquidity ratio has a similar effect by increasing the liquidity shortage in the bad state,  $(c - b)n$ , and hence by causing a larger supply of risky assets to the market. Lower asset prices, by contrast, induces more fire sales by banks due to the downward-sloping supply curve. This result is formalized in Lemma 2.

**Lemma 2.** *The fraction of assets sold at  $t = 1$ ,  $1 - \gamma(n, b)$ , is increasing in  $n$  and decreasing in  $b$ .*

Together lemmas 1 and 2 imply that a higher initial investment in the risky investment by one bank or a lower liquidity ratio creates negative externalities for other banks by making financial crises more severe (that is, via lower asset prices according to Lemma 1) and more costly (that is, more fire sales according to Lemma 2).

### 3.2 Competitive equilibrium

As a benchmark, we first characterize the equilibrium when there is no regulation on banks. At the initial period each bank  $i$  chooses the amount of investment in the risky asset,  $n_i$ , and the liquidity ratio,  $b_i$ , to maximize its expected profits:

$$\max_{n_i, b_i} \Pi_i(n_i, b_i) = (1 - q)\{R + b_i\}n_i + q\{I(b_i < c)R\gamma_i + I(b_i \geq c)[R + b_i - c]\}n_i - D(n_i(1 + b_i)) \quad (7)$$

subject to the budget constraint at  $t = 0$ ,  $0 \leq (1 + b_i)n_i \leq e$ .  $D(n_i(1 + b_i)) = n_i(1 + b_i) + \Phi(n_i(1 + b_i))$  is the sum of the initial cost of funds and operational costs of a bank. Because we assume that  $\Phi(\cdot)$  is convex, it follows that  $D(\cdot)$  is convex as well, that is,  $D'(\cdot) > 0$  and  $D''(\cdot) > 0$ .

Whether fire sales take place in the competitive equilibrium or not depends on the initial liquidity ratios of bank. If banks fully insure themselves against the fire sale risk, that is, if they choose  $b_i \geq c$  for all  $i \in [0, 1]$  at  $t = 0$ , then fire sales in the bad state are avoided completely. However, if banks purchase less than full insurance, that is, if  $b_i < c$ , then fire sales exist. The following lemma shows that in the competitive equilibrium banks optimally choose less than full insurance and hence fire sales take place.

**Proposition 1.** *Banks take fire sale risk in equilibrium, that is,  $b_i < c$  for all banks.*

*Proof.* It is straightforward to show that banks will never carry excess liquidity in equilibrium, that is,  $b_i > c$ . This is because when  $b_i > c$  the liquid assets in excess of the shock,  $(b_i - c)n$ , have no use even in the bad state; the expected return on liquid assets is equal to one and dominated by the expected return on the risky asset,  $R - cq$ , by the *Technology* assumption. Therefore, for contradiction assume that  $b_i = c$ . Corresponding first order conditions of bank's problem given by (7) with respect to  $n_i$  and  $b_i$  are respectively:

$$(1 - q)(R + b_i) + qR = D'(n_i(1 + b_i))(1 + b_i), \quad (8)$$

$$(1 - q)n_i + qn_i = D'(n_i(1 + b_i))n_i. \quad (9)$$

The last equation implies that  $D'(n_i(1 + b_i)) = 1$ . Substitute this into the first equation to obtain  $R + (1 - q)b_i = 1 + b_i$ . Now using  $b_i = c$  gives  $R + (1 - q)c = 1 + c$ , which contradicts with the *Technology* assumption, that is,  $R > 1 + cq$ . Therefore, we must have  $b_i < c$  for all  $i \in [0, 1]$ .  $\square$

Even though both the amount ( $c$ ) and frequency ( $q$ ) of the aggregate liquidity shock are exogenous in the model, whether and how much fire sale takes place is endogenously determined. In principle, banks can insure themselves perfectly against the liquidity shock by holding sufficient amount of liquidity, that is,  $b_i = c$ . However, Proposition 1 shows that perfect insurance is never optimal and banks will take some amount of fire sale risk, that is, they choose  $b_i < c$ . The main issue is whether banks take the socially optimal amount of fire-sale risk or not. We will show that because banks do not internalize the pecuniary externalities they end up taking too much risk. This is why bank regulation is needed in this economy.

Proposition 1 allows us to focus on the imperfect insurance case, that is,  $b_i < c$ . We start by solving for the competitive equilibrium price in this case using the first order conditions the banks' problem, given by (7), when  $b_i < c$ . Furthermore, we show in the next proposition that the analytical solution for  $P$  is independent of the functional form of the outside investors' demand and the operational cost of banks. However, in order to solve for equilibrium investment levels and liquidity ratios, we need to make some functional form assumptions.

**Proposition 2.** *The competitive equilibrium price of assets is given by:*

$$P = \frac{qR(1+c)}{R-1+q}. \quad (10)$$

*The equilibrium price,  $P$ , is increasing in the probability of the liquidity shock,  $q$ , and the size of the shock,  $c$ , but decreasing in the return on the risky assets,  $R$ .*

### 3.2.1 A closed-form solution for the competitive equilibrium

In order to obtain closed form solutions for the equilibrium values of  $n$  and  $b$ , we need to make functional form assumptions for outside investors' production technology,  $F$ , and the operational cost of banks,  $\Phi$ . Suppose that the operational costs of a bank are given by  $\Phi(x) = dx^2$ , and hence  $\Phi'(\cdot)$  is increasing, that is,  $\Phi'(x) = 2dx$ . On the demand side, suppose that the return function of outside investors is given by  $F(y) = R \ln(1+y)$ . It is easy to verify that this function satisfies the *Concavity*, *Elasticity* and *Regularity* assumptions. In Section 7.1 in the Appendix solve for the competitive equilibrium investment level and liquidity ratios as follows:

$$n = \frac{\tau}{\tau+1} \frac{q(\tau+1) + 2dR}{2d(1+c)}, \quad b = \frac{cq(\tau+1) - 2dR}{q(\tau+1) + 2dR}, \quad (11)$$

where

$$\tau = \frac{R}{P} - 1 = \frac{R-1+q}{q(1+c)} - 1.$$

Proposition 3 presents the comparative statics of the competitive equilibrium liquid and risky investment levels above with respect to model parameters.



**Proposition 3.** *The comparative statics for the competitive equilibrium risky investment level,  $n$ , and liquidity ratio,  $b$ , are as follows:*

1. *The risky investment level ( $n$ ), is increasing in the return on the risky asset, ( $R$ ), and decreasing in the size of the liquidity shock, ( $c$ ), marginal cost of funds ( $d$ ), and the probability of the bad state, ( $q$ ).*
2. *The liquidity ratio ( $b$ ), is increasing in the size of the liquidity shock ( $c$ ), the return to the risky asset, ( $R$ ), and the probability of the bad state, ( $q$ ), and decreasing in the marginal cost of funds, ( $d$ ).*

Propositions 3 shows that  $b$  and  $n$  move in the same direction as response to following parameters:  $R$  and  $d$ , while they move in opposite directions as response to  $c$  and  $q$ . This is intuitive since  $cq$  is the expected value of the liquidity need at the interim period. As the expected liquidity need increases, the bank holds more liquidity and less risky assets. Of course that does not say whether the bank increases its liquidity ratio sufficiently from a socially optimal perspective.

### 3.3 Partial Regulation: Regulating only capital ratios

In this section we consider the problem of a regulator who chooses the optimal level of risky investment  $n \geq 0$  at  $t = 0$  to maximize the net expected social welfare, but allows banks to freely choose their liquidity ratio ( $b_i$ ). In the next section, we will show that the optimal risky investment level in this case is lower than the privately optimal level of risky investment that banks choose when there is no regulation. As a result, the regulator can implement the optimal level by introducing it as a regulatory upper limit on domestic banks' risky investment level. We consider this case to mimic the regulatory framework in the pre-Basel III period, which predominantly focused on capital adequacy requirements. We first analyze the problem of a representative bank for a given regulatory investment level,  $n$ . The bank chooses the liquidity ratio ( $b_i$ ) to maximize its expected profits, and hence, the problem of the bank is as follows:

$$\max_{b_i} \Pi_i(n, b_i) = (1 - q)\{R + b_i\}n + qR\gamma_i n - D(n(1 + b_i)) \quad (12)$$

The first order condition of banks' problem (12) with respect to  $b_i$  is:

$$1 - q + qR\frac{1}{P} = D'(n(1 + b_i)) \quad (13)$$

From this first order condition, we can obtain banks' (implicit) reaction function to the regulatory investment level that is, the optimal liquidity ratio,  $b_i$ , that banks choose for each given risky investment level  $n$  is as follows:

$$b_i(n) = \frac{D'^{-1}(1 - q + q\frac{R}{P})}{n} - 1 \quad (14)$$

The regulator takes this reaction function into account while choosing the optimal risky investment level to maximize the expected social welfare:

$$\max_n W(n) = (1 - q)\{R + b(n)\}n + qR\gamma n - D((1 + b(n))n) + e. \quad (15)$$

The social welfare is the sum of expected bank profits and the expected utility of a representative consumer.<sup>16</sup> Note that the later is equal to  $(e - Li) + Li = e$ : Depositors are risk-neutral and they consume  $e - L_i$  units at  $t = 0$ , and  $L_i$  units at  $t = 2$  in both states because deposits are safe and produce a unit return. Unlike an unconstrained social planner, the regulator is subject to the same constraints as the private agents. In particular, he takes the collateral constraints of banks in the bad state as given. The only difference of this regulatory objective function from banks' problem given by (7) is that the regulator takes into account the effect of initial risky investment level on the price of assets in the bad state. The optimal risky investment level in this case will be determined by the following first order conditions of the regulator's problem above with respect to  $n$ :

$$(1 - q)\{R + b(n) + nb'(n)\} + qR\{\gamma + n\frac{d\gamma}{dn}\} = D'(n(1 + b))\{1 + b(n) + nb'(n)\} \quad (16)$$

Changing  $n$  has an indirect effect on the asset price in the bad state in addition to its direct effect because banks change their liquidity ratios as a response when  $n$  changes which then changes the price of assets in the equilibrium. The main question in this case is how do banks respond to a tightening of capital regulations. In other words, do banks increase or decrease their liquidity ratios,  $b_i$ , when the regulator reduces the risky investment level,  $n$ ? Proposition 4 answers this question.

**Proposition 4.** *Let the operational cost of a bank be given by  $\Phi(x) = dx^2$ . Then for any technology for outside investors  $F$  that satisfies the *Concavity*, *Elasticity* and *Regularity* assumptions with  $F'(0) = R$ , banks decrease their liquidity ratio as the regulator tightens capital requirements, that is,  $b'(n) \geq 0$ .*

Proposition 4 shows that banks reduce their liquidity ratios as the regulator tightens the risky investment level. The regulator attempts to correct excessive risk taking of banks by reducing the the risky investment of banks by introducing a risk-weighted capital ratio requirement. However, because this regulation prevents banks from reaching their privately optimal level of risk, they react by reducing their liquidity ratios. In other words, banks undermine the purpose of capital regulations by carrying less liquid portfolios. It would not be surprising to observe banks to hold a lower amount of liquid assets when they are asked by the regulator to decrease their risky asset holdings. However, what Proposition 4 shows goes beyond that: Banks decrease their liquidity

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<sup>16</sup>We assume that the regulator does not consider the welfare of outside investors. This can be justified when the regulator is solely concerned with the well being of banks and their depositors. However, the results of the paper are robust to relaxing this assumption, as discussed in Section 5.

ratios as well when the the regulator limits risky investment level, that is, banks hoard less liquidity per unit of risky asset.

The intuition of the proof is as follows: The marginal return to the liquidity ratio,  $b_i$ , is  $(1 - q)n_i + qR\frac{1}{P}n_i$ , and it is decreasing in the fire sale price  $P$ . Each unit of liquidity holding per risky assets becomes more valuable as the fire-sale price decreases. From Lemma 1 we know that the fire sale price is decreasing in the aggregate amount of risky investment  $n$ . As the fire sale price tends to increase due to the regulation limiting the amount of risky investment,  $n$ , liquidity hoarding becomes less attractive for the banks. The benefit of holding liquidity is to be less exposed to fire sale risk. A higher fire sale price means that the risk is less costly, thus it is optimal for banks to take more of that risk.

Note the role of rational expectations in generating the above mentioned relationship between capital regulation and liquidity ratios. Because the regulation applies to every bank, banks correctly forecast the fire-sale price to be higher due to less risky investment in the banking system, and they optimally decrease their liquidity ratios. In a sense, Proposition 4 reveals an unintended consequence of capital regulation when it is applied in isolation. If the financial system is now more stable, that is higher fire-sale prices and fewer fire sales, banks' incentive to hoard liquidity is smaller.

Proposition 4 implies that for any elastic, log-concave demand function with the intercept  $F'(0) = R$ , the result  $b'(n) \geq 0$  holds. However, the condition is actually looser than that because as we have discussed above the *Regularity* assumption is weaker than log-concavity. In the appendix, we show that it is even possible relax this constraint further and obtain this result with weak inequality, that is,  $F'(0) \leq R$ , if we make the following assumption on outside investors technology:

$$R < \frac{F'(F' + yF'')}{F' + 2yF''} \quad (17)$$

The proof provides a sufficient condition by showing that there is strategic complementarity between the regulatory risky investment level,  $n$ , and the liquidity ratio,  $b_i$ , for each bank. We use the monotone comparative statics techniques outlined in Vives (2001) to show that banks' profit function exhibits increasing differences between  $n$  and  $b_i$  in the partial regulation case.

### 3.3.1 A closed-form solution for the partial regulation case

Using the functional form assumptions discussed above, in Section 7.2 in the Appendix we derive the solution to the partial regulation case as follows:

$$b^* = \frac{cq(\tau^* + 1) - 2dR}{q(\tau^* + 1) + 2dR}, \quad n^* = \frac{\tau^*}{\tau^* + 1} \frac{q(\tau^* + 1) + 2dR}{2d(1 + c)} \quad (18)$$

where  $\tau^* \equiv R/P^* - 1$ , and  $P^*$  is the only real (positive) root of the cubic equation:<sup>17</sup>

$$2d\sigma P^{*3} + [qR\sigma - 2d\beta]P^* - q\beta = 0, \quad (19)$$

and  $\sigma \equiv (R - 1 + q)/(qR)$  and  $\beta \equiv R(1 + c)$  are defined in terms of the parameters of the model.

### 3.4 Complete Regulation: Regulating both capital and liquidity ratios

Consider the problem of a regulator who optimally chooses both the risky investment level,  $n$ , and the liquidity ratio,  $b$ , at  $t = 0$  to maximize the net expected social welfare:

$$\max_{n,b} W_i(n_i, b_i) = (1 - q)\{R + b\}n + q\{I(b < c)R\gamma + I(b \geq c)[R + b - c]\}n - D(n(1 + b)) + e, \quad (20)$$

subject to the budget constraint at  $t = 0$ ,  $0 \leq (1 + b)n \leq e$ . The social welfare is equal to the sum of expected bank profits and the expected utility of a representative consumer, which is equal to  $e$ . In the next section we will show that socially optimal risky investment level is lower than privately optimal level and the socially optimal liquidity ratio is higher than the privately optimal ratio due to the existence of pecuniary externalities. Therefore, the regulator can implement the optimal values by imposing a minimum ratio of liquidity holding for each bank as a fraction of its risky asset ( $b_i \geq b$ ) and a maximum level of risky investment ( $n_i \leq n$ ), which corresponds to a minimum risk weighted capital ratio, that is  $E/n_i \geq E/n$ .

The first question is would the regulator avoid fire sales completely by setting  $b \geq c$ . The next proposition shows that the answer is no, that is, even the regulator would optimally expose the banking sector to some amount of fire sale risk.

**Proposition 5.** *It is socially optimal to take fire sale risk, that is, the socially optimal liquidity ratio satisfies  $b < c$ .*

The proposition states that it is socially optimal to expose the banking sector to fire sale risk. In other words the full insurance is not socially optimal. A higher liquidity ratio decreases fire sales by decreasing the liquidity shortage of each bank (micro-prudential) and also by increasing the price of the risky asset (macro-prudential). In other words, holding liquidity makes each bank less exposed to the fire sale risk and the fire sales are less severe thanks to higher fire sale prices. However the marginal benefit decreases with the amount of liquidity, both the liquidity at the bank and the aggregate level. On the other hand the foregone profit from not investing in risky asset constitutes the cost of liquidity holding. The social planner weighs the opportunity cost against the micro and macro-prudential roles of liquidity in the bad state and determines the optimal amount of fire sale risk to take.

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<sup>17</sup>The solution to this cubic equation can easily be obtained using Vieta's substitution. However, we will not show the solution here to save space, and also because we do not use the explicit solution in any of the proofs.

Proposition 5 allows us to focus on the case  $b < c$  when analyzing the regulators problem given by (20). Corresponding first order conditions with respect to  $n$  and  $b$  are respectively:

$$(1 - q)(R + b) + qR \left\{ \gamma + \frac{(c - b)}{P^2} \frac{\partial P}{\partial n} n \right\} = D'(n(1 + b))(1 + b), \quad (21)$$

$$(1 - q)n + qR \left\{ \frac{1}{P} + \frac{(c - b)}{P^2} \frac{\partial P}{\partial b} \right\} n = D'(n(1 + b))n, \quad (22)$$

where  $\gamma = 1 - \frac{c-b}{P}$ . These first order conditions are different from the first order conditions of banks' problem in Section 3.2 in that the regulator takes into account of changing initial risky investment level and liquidity ratio affects the price of assets,  $P$ , and the fraction of asses sold to outside investors,  $1 - \gamma$ . Unlike an unconstrained social planner, however, the regulator is subject to the same market constraints as the private agents. In particular, he takes the collateral constraints of banks in the bad state as given.

### 3.4.1 A closed-form solution for the complete regulation case

In order to solve for the equilibrium outcomes we employ the functional form assumptions for outside investors' technology and banks' operational cost mentioned above. As shown in Section 7.3 in the Appendix, closed-form solutions for  $n^{**}$ ,  $b^{**}$  and  $P^{**}$  can be obtained as follows:

$$n^{**} = \frac{2dr\tau^{**} + q\tau^{**}(\tau^{**} + 1)(\tau^{**} + 2)}{2d(1 + c)(\tau^{**} + 1)} \quad (23)$$

$$b^{**} = \frac{cq(\tau^{**} + 1)(\tau^{**} + 2) - 2dR}{2dR + q(\tau^{**} + 1)(\tau^{**} + 2)} \quad (24)$$

$$P^{**} = \sqrt{\frac{qR^2(1 + c)}{R - 1 + q}}, \quad (25)$$

where

$$\tau^{**} = \frac{R}{P^{**}} - 1 = \sqrt{\frac{R - 1 + q}{q(1 + c)}} - 1 \quad (26)$$

## 4 Comparison of equilibrium outcomes

In this section, we compare and contrast the equilibrium outcomes (level of risky assets, liquidity ratios, asset prices and the amount of fire sales) in three different settings: decentralized equilibrium without any regulation, partially regulated case in which there is only capital regulation to limit risky investment level, and the complete regulation case in which both the capital and liquidity ratios are regulated. We use the closed form solutions of equilibrium outcomes to do the comparison, which were obtained when the operational costs of a bank are given by  $\Phi(x) = dx^2$ , and the production technology of outside investors is given by  $F(y) = R \ln(1 + y)$ . Proposition 6 summarizes

the results and we discuss these results with reference to Figures 3 to 7.

**Proposition 6.** *Risky investment levels, liquid asset holdings, and financial stability measures under competitive equilibrium  $(n, b, 1 - \gamma, P)$ , partial regulation equilibrium  $(n^*, b^*, 1 - \gamma^*, P^*)$ , and complete regulation equilibrium  $(n^{**}, b^{**}, 1 - \gamma^{**}, P^{**})$  compare as follows:*

1. *Risky investment levels:  $n > n^{**} > n^*$*
2. *Liquidity ratios:  $b^{**} > b > b^*$*
3. *Financial stability measures*
  - (a) *Fraction of assets sold:  $1 - \gamma > 1 - \gamma^* > 1 - \gamma^{**}$*
  - (b) *Total fire sales:  $(1 - \gamma)n > (1 - \gamma^*)n^* > (1 - \gamma^{**})n^{**}$*
  - (c) *Price of assets in the bad state:  $P^{**} > P^* > P$*

It is not surprising that  $n > n^{**}$  and  $b^{**} > b$ , that is, in competitive equilibrium banks hold more risky asset and lower liquidity ratios compared to the socially optimum levels, for banks do not internalize the fire sale externality. However, what is more interesting here is that the socially optimum level of risky investment,  $n^{**}$ , is higher than the risky investment in the partial regulation case, that is,  $n^{**} > n^*$ . In other words, the minimum capital ratio is inefficiently high under partial regulation as also shown in Figure 3. When the capital requirement is the only tool that the social planner or the regulator has, then it is used excessively. This result can be better explained when it is considered together with the comparison of liquidity holdings in these two cases. As shown in Figure 4 as well, the socially optimum level of liquidity,  $b^{**}$ , is higher than the liquidity chosen by banks under the partial regulation,  $b^*$ . Having more liquidity allows holding more risky assets without increasing the fire sale risk. Therefore, the socially optimal choice is to hold a higher level of risky investment that is supported with greater liquidity ratios. For example, consider a country that is transitioning from the partial regulation to the complete regulation by imposing new liquidity rules in addition to the existing capital rules. In part, this can be compared to moving from Basel II to Basel III regulatory approach.<sup>18</sup> Assuming that the capital regulation had been set optimally during the Basel II period, with the addition of liquidity requirements the capital requirements can be relaxed. Proposition 4 provides further clarification because it shows that banks' response to stricter capital regulation is to hold lower liquidity ratios. In the partial regulation case, in which the regulator does not have the sufficient number of tools, the regulator predicts that the banks will have lower liquidity ratios, and hence tightens the capital requirements further.

How effective is the capital regulation in addressing financial instability caused by fire sales when it is not accompanied by liquidity requirements? In order to answer this question, we can compare

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<sup>18</sup>To be more precise, liquidity requirements in Basel III are different than the simple liquidity requirements that we consider in this paper.

and contrast the measures of financial instability across the two different regulatory regimes: asset prices and fire sales. A lower price of the risky asset in the bad state implies that the fire sales are more severe and the externality has a strong presence in the economy. Figure 5 shows that the fire sale price under partial regulation is not significantly different from the competitive equilibrium price, but it is far below the socially optimal price level. We reach a similar conclusion when we consider other measures of the severity of a crisis. Figure 6 shows the fraction of risky asset that must be sold to withstand the liquidity shock. These fractions are almost the same under the partial regulation and the unregulated environments, and they both are much higher than the fraction under the complete regulation. From this point of view, the partial regulation has almost no effect on limiting the fire sales. When we look at the total amount of risky asset sold under fire sales, as shown by Figure 7, the message is the same:<sup>19</sup> Capital regulation has only a mild effect in terms of financial stability when it is not accompanied with liquidity requirements. The biggest improvement in terms of financial stability is achieved when both capital and liquidity are regulated. That is because, when the capital is regulated but liquidity is not, banks undermine the regulation by decreasing their liquidity ratios.

Minimum capital rules may serve to several other purposes such as countering moral hazard problems generated by the existence of limited liability and deposit insurance, which we do not analyze in this paper. However, what we show in this paper is that, for the purpose of fire sales related financial instability, capital regulations are not effective if they are not combined with liquidity requirements. In relation to the regulation of liquidity, the important point is that banks hold a smaller fraction of their asset in terms of liquid items compared to unregulated case (competitive equilibrium). Because  $b > b^*$ , once the liquidity shock hits, banks will run out of liquidity earlier under the partially regulated case. Although banks have a smaller liquidity buffer in the bad state under partial regulation, the capital regulation still achieves an improvement in financial stability by reducing the aggregate risky investment level. Both in terms of fraction of the total risky asset and in terms of the total risky asset sold under fire sales, the amount of fire sales is smaller under partial regulation compared to the competitive equilibrium.

The level of risky investment or capital ratios alone are not very informative about the potential fire sales; a bank with a lower level of risky investment may end up conducting a larger fire sale compared to a bank with higher level of risky investment. This can happen if a lower liquidity ratio accompanies the low level of risky investment, as in the case of partial regulation.

We end this section by comparing the bank size across three different regimes in the following proposition, and discuss the implications of this result for simple leverage ratio regulation.

**Proposition 7.** *Bank balance sheet sizes across different regimes are as follows:*

$$n(1 + b) = n^{**}(1 + b^{**}) > n^*(1 + b^*) \quad (27)$$

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<sup>19</sup>The difference between Figures 6 and 7 is due to different levels of risky investment.

Proposition 7 provides an interesting result on the regulation of simple leverage ratio, which can be defined as  $\frac{E}{n(1+b)}$  in this setup. Proposition 7 shows that the optimal simple leverage ratio is the same under complete regulation and competitive (unregulated) cases. That means, in our setup, starting from the competitive equilibrium, it is possible to obtain the *constrained social optimum* by reallocating funds from risky asset to liquid asset, without affecting the balance sheet size. However, Proposition 6 has shown the difference between the two cases in terms of financial stability is dramatic. This suggests that from a financial stability perspective what is important is not the size of banks' balance sheet but the composition of it. Therefore, we can conclude that in the current setup, a leverage regulation applied in isolation would be completely ineffective. However, the leverage regulation once combined either with a liquidity ratio requirement or with an risk-weighted capital regulation is sufficient to replicate the *constrained social optimum*.

## 5 Discussion of assumptions

In this section we show that the results of the paper are robust to some changes in the modeling environment. First, we consider including outside investors in the regulator's objective function. So far the regulator cared only about the welfare of banks and their depositors. This can be justified by the institutional arrangements. For instance, the regulator might be only responsible for the deposit collection institutions and therefore does not consider the well-being of non-depository institutions. However, for the sake of completeness, in this section we include the profit of outside investors into the aggregate social welfare and show that the results are invariant to this change. Second, we consider deposit markets that have monopolistic competition and endogenize the deposit rate while assuming that banks have limited liability. In previous sections we assumed that the bank equity is sufficiently large to prevent banks from default in the bad state and, as a result, each bank could raise deposits from consumers at net zero interest rate. This new setup allows default in equilibrium. Third, we introduce deposit insurance and limited liability together. We did not need the limited liability or deposit insurance assumptions in the basic setup because there was no default in equilibrium.

We show that the qualitative results of the paper do not change under these different modeling environments. This is because the constrained inefficiency of the decentralized equilibrium, and hence the justification for capital regulations and liquidity regulations, do not depend on a moral hazard problem created by the existence of deposit insurance or limited liability for bank owners. Instead, the inefficiency depends purely on the existence of pecuniary externalities under incomplete markets. In other words, the inefficiency driven by fire-sale externalities would prevail in a narrow banking system, in which the banks are all financed by nothing but equity capital.

Last, we show that the aggregate nature of the liquidity shock is not material for the mechanism or the conclusions of the model. We allow the liquidity shock to be idiosyncratic rather than being aggregate and show that this setup is isomorphic to the aggregate shock case with a lower size of



liquidity shock.

## 5.1 Including outside investors in the welfare function

In previous sections the regulator was concerned only about the welfare of bankers and depositors, while ignoring the welfare of global investors. In this section, we discuss relaxing this assumption and incorporating the profits of global investors into the regulator's welfare function. In particular, we define the social welfare function as the sum of the expected bank profits, expected utility of a representative consumer and a global investor with equal weights.<sup>20</sup>

We obtain analytically solutions for the partial and complete regulation cases after including outside investors in the welfare calculations under the same functional assumptions as in the main section. Then, using numerical examples under a wide region of parameters, we show that our results continue to hold in this more general setup.<sup>21</sup> This is because outside investors are less efficient than banks and in the bad state some of the risky assets are allocated from more efficient sector (banking) to a less efficient sector (outside investors) via fire-sales. The greater the amount of fire-sales, the more assets are re-allocated from banks to the outside investors, thus the greater is the ex-post inefficiency. The fire sale price, as well as the amount of fire-sales, is a sufficient statistic for ex-post inefficiency. Therefore, including the well-being of outside investors does not tilt the regulator's objective function. It only makes fire sales socially less costly, because the regulator no longer treats the assets that are transferred to outside investors during fire sales as disappeared from the system. However, this does not change the fact that fire sales are costly and banks get exposed to the fire sale risk sub-optimally by holding too much risky assets and too low liquidity ratios.

## 5.2 Endogenizing the deposit rate

In previous sections, we assumed that the bank equity is sufficiently large to prevent banks from default in the bad state and, as a result, each bank could raise deposits from consumers at net zero interest rate. Now, instead, suppose that each bank is a local monopsony in the deposit market and there is limited liability for banks.<sup>22</sup> Consider the decentralized equilibrium without regulation in this setup. At the initial period, bank  $i$  will choose the amount to invest in the risky asset,  $n_i$ , liquidity ratio,  $b_i$ , and the interest rate on the deposits,  $r_i$ , to maximize the net expected profits:

$$\max_{r_i, n_i, b_i} (1 - q)[(R + b_i)n_i - r_i L_i] + q \max\{R\gamma n_i - r_i L_i, 0\} - E - \Phi((1 + b_i)n_i), \quad (28)$$

<sup>20</sup>The risk neutrality of all agents allows us to write the welfare function in this way.

<sup>21</sup>These results are not included here but are available upon request from the authors.

<sup>22</sup>We restrict attention to deposit contracts that are in the form of simple debt contracts. Debt contracts can be justified by assuming that depositors can observe banks' asset returns only at a cost. According to [Townsend \(1979\)](#), in the case of costly state verification, debt contracts are optimal.

subject to

$$(1 - q)r_i L_i + q \min\{R\gamma n_i, r_i L_i\} \geq L_i \quad (IR), \quad (29)$$

where  $\gamma = 1 - (c - b_i)/P$  is the fraction of assets retained by banks at  $t = 1$  after fire sales (which, as before, banks take as given) and  $L_i$  is the amount of deposits. The bank has to satisfy the individual rationality (IR) constraint of consumers given in (29): Expected return to deposits must be greater than the initial deposit of a consumer,  $L_i = (1 + b_i)n_i - E$ . Each consumer receives a gross return of  $r_i L_i$  in the good state, which happens with probability  $1 - q$ . In the bad state, which arises with probability  $q$ , he obtains the minimum of the promised payment,  $r_i L_i$ , and the returns available to the bank after fire sales,  $R\gamma n_i$ . If  $R\gamma n_i < r_i L_i$ , the bank defaults in the bad state.

First, consider the choice of optimal  $r_i$  for a given investment level  $n_i$ , and liquidity ratio,  $b_i$ . Because each bank is a local monopsony, the interest on deposit contracts needs to be just enough to induce risk-neutral consumers to deposit their endowments with them. In technical terms, the individual rationality condition for consumers binds. If the returns in the bad state are such that  $R\gamma n_i > L_i$ , then banks can optimally set  $r_i^* = 1$ . This reduces to the exactly same case we considered in the basic model setup because deposits are now safe. However, if  $R\gamma n_i < L_i$ , banks have to offer a positive net interest rate to consumers in the good state to compensate for their losses in the bad state. For the IR condition of consumers to be satisfied,  $r_i$  has to be such that

$$r_i \geq \frac{L_i - qR\gamma n_i}{(1 - q)L_i} \equiv r_i^*. \quad (30)$$

This can be seen by rearranging the IR condition (29) and noting that  $\min\{R\gamma n_i, r_i L_i\} = R\gamma n_i$ . In order to maximize profits, banks set  $r_i = r_i^*$ .<sup>23</sup> Now, we can substitute the optimal  $r_i^*$  back into the banks' objective function (28) and simplify to obtain:

$$\max_{n_i, b_i} (1 - q)(R + b_i)n_i + qR\gamma n_i - (1 + b_i)n_i - \Phi((1 + b_i)n_i), \quad (31)$$

where we use that  $L_i + E = (1 + b_i)n_i$  and  $\max\{R\gamma n_i - r_i L_i, 0\} = 0$  because  $R\gamma n_i - L_i < 0$  in equilibrium as argued above. Note that this problem is the same as the banks' problem that we have considered in Section 3.2. Therefore, the optimal level of risky investment and liquidity ratio in the decentralized equilibrium are the same as before. We can define the social welfare in this setup as the sum of the bank profits, given by (28), and the expected utility of a representative consumer. After substituting for equilibrium interest rate  $r_i^*$  into the left-hand side of (29), the

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<sup>23</sup>This setup requires the assumption that depositors can perfectly observe the equilibrium level of investment ( $n_i$ ), the fraction of assets sold ( $1 - \gamma$ ), and the price of assets ( $P$ ). However, we show in Section 5.3 that the results of the paper do not change when we change the environment by introducing deposit insurance, which does not require this perfect observation assumption.

latter can be obtained as:

$$(e - L_i) + (1 - q)L_i \frac{L_i - qR\gamma n_i}{(1 - q)L_i} + qR\gamma n_i = e, \quad (32)$$

where we used that  $\min\{R\gamma n_i, r_i L_i\} = R\gamma n_i$  in equilibrium. Therefore, we obtain that the social welfare function is the same as before. This implies that all results in the paper continue to hold in this setup.

### 5.3 Deposit insurance and limited liability

The deposit insurance can be introduced to the model with a slight modification. Suppose that the regulator (or a separate insurance agency) runs a domestic deposit insurance fund that is fairly priced. In particular, banks pay deposit insurance fees in good times, and in exchange the deposit insurance covers any deficit between the banks' return and the promised payments to depositors in bad times. We assume that each bank is a local monopsony in the deposit market as above. Because of the deposit insurance, banks maximize profits by offering consumers a net zero interest rate. As a result, consumers inelastically supply their endowments to banks at the initial period. Let  $\tau_i$  be the fee that banks pay to the deposit insurance in good times per unit of deposits. Banks' problem changes as follows:

$$\max_{n_i, b_i} (1 - q)[(R + b_i)n_i - L_i - \tau_i L_i] + q \max\{R\gamma n_i - L_i, 0\} - E - \Phi((1 + b_i)n_i), \quad (33)$$

The fair pricing of the deposit insurance requires  $(1 - q)\tau_i L_i = q \max\{L_i - R\gamma n_i, 0\}$ . Substituting this fair value back into banks' problem above and noting that banks default in the bad state, that is,  $\max\{R\gamma n_i - L_i, 0\} = 0$ , gives:

$$\max_{n_i, b_i} (1 - q)(R + b_i)n_i + qR\gamma n_i - E - L_i - \Phi(n_i(1 + b_i)). \quad (34)$$

Using  $L_i = (1 + b_i)n_i - E$  this can be written as:

$$\max_{n_i, b_i} (1 - q)(R + b_i)n_i + qR\gamma n_i - n_i(1 + b_i) - \Phi(n_i(1 + b_i)) \quad (35)$$

The problem of banks given by (35) is the same as in Section 3.2. Therefore, the optimal level of investment and liquidity ratio in the decentralized equilibrium without regulation remain the same. Note that the representative depositor consumes  $e - L_i$  at the initial period and  $L_i$  in the final period, in both states. As a result, the depositor's expected utility at  $t = 0$  is  $e - L_i + L_i = e$ . As a result, the social welfare function is also exactly the same as before. To conclude, all of the results in the paper are robust to adding deposit insurance and limited liability for banks in the model.

## 5.4 Idiosyncratic liquidity shocks

In the basic model the liquidity shock has been an aggregate one in nature, similar to [Lorenzoni \(2008\)](#). In this section we show that the aggregate nature of liquidity shock is without loss of generality and our results do not change qualitatively if we allow idiosyncratic liquidity shocks. In this more general setup, liquidity shocks hit only a fraction of the banks. Thus, banks are ex-post heterogeneous in terms of their liquidity needs. Banks that receive the liquidity shock need funding while others are left with excess liquidity. Banks with excess liquidity can use these resources to buy the risky assets from the distressed banks, potentially at fire sale prices.<sup>24</sup> The amount of risky asset that can be bought with the liquid holdings of a shock-free bank is  $\frac{b_i n_i}{P}$ . First, we analyze the case conditional on the liquidity shock, but without knowing which banks get the shock. We assume that conditional on being in the bad state, the probability of being hit with a liquidity shock is  $\lambda$  for each bank. Hence, by the law of large numbers, a fraction  $\lambda$  of banks is hit by the liquidity shock in the bad state. The expected profit of a bank before the realization of which banks receive the shock, conditional on the bad state, is:  $\lambda R \gamma_i n_i + (1 - \lambda)(n_i + \frac{b_i n_i}{P})R$ . The first term,  $\lambda R \gamma_i n_i$ , is the return from remaining risky assets after fire sales, multiplied by the probability of receiving the liquidity shock,  $\lambda$ . The amount of remaining risky assets after fire sales is denoted by  $\gamma_i$  for bank  $i$  as in the benchmark model. We substitute for  $\gamma_i$  and rewrite the expected profit conditional on the bad state as follows

$$\begin{aligned}
 E[\pi_i | q = 1] &= \lambda R \left(1 - \frac{c - b_i}{P}\right) n_i + (1 - \lambda) n_i R + (1 - \lambda) \frac{b_i n_i}{P} R \\
 &= \lambda R n_i - \frac{\lambda R c n_i}{P} + \lambda R \frac{b_i n_i}{P} + (1 - \lambda) R n_i + \frac{b_i n_i}{P} R - \lambda \frac{b_i n_i}{P} R \\
 &= R n_i + R \left(\frac{b_i - c \lambda}{P}\right) n_i \\
 &= R \left(1 - \frac{c \lambda - b_i}{P}\right) n_i \\
 &= R \tilde{\gamma}_i n_i
 \end{aligned}$$

where  $\tilde{\gamma}_i = 1 - \frac{c \lambda - b_i}{P}$  and it is similar to  $\gamma_i$  in the basic setup when the size of the liquidity shock,  $c$ , is replaced with  $c \lambda$ . In this setup when we set  $\lambda$  equal 1 we are back to our benchmark case. Thus allowing  $\lambda$  to be between zero and one provides us a more general model. In order to write the expected profits of banks at  $t = 0$  in this more general setup, we just have to note that the economy ends up in the bad state only with probability  $q$  and obtain the returns derived above, while in good times, which arise with probability  $1 - q$ , the returns are the same as in the benchmark case:

$$E[\pi_i] = (1 - q)(R + b_i)n_i + qR\tilde{\gamma}_i n_i$$

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<sup>24</sup>In principle it is possible that the amount of excess liquidity in the banking system exceeds the liquidity need of the shock receiving banks. At the end of this subsection we explain why this does not arise in equilibrium.

where  $\tilde{\gamma}_i = 1 - \frac{c\lambda - b_i}{P}$ .

Compared to the benchmark case, the only difference in banks' expected profit at  $t = 0$  is that  $c$  is replaced with  $c\lambda$ . For completeness we conclude by writing the demand and supply functions in this more general case. The aggregate liquidity need in the bad state is  $\lambda(c - b)n$  and the liquidity supply is  $(1 - \lambda)bn + PQ^d(P)$ . Equating demand and supply yields  $\lambda(c - b)n = (1 - \lambda)bn + PQ^d(P)$ , and simplifying reduces this market clearing condition to  $(\lambda c - b)n = PQ^d(P)$ . Compared to the market clearing condition in the original setup the only difference is again  $c$  being replaced with  $\lambda c$ . Thus, in this new setup if we relabel  $\lambda c = \tilde{c}$  we are back to the our original setup where  $c$  is replaced with  $\tilde{c}$ .

Is it possible to have no fire sales in the bad state in this setup? This would be possible if the liquid assets in the hands of shock-free banks are in excess of the liquidity need of shock-receiving banks, so that the risky assets are traded within the banking system without needing to sell to outside investors. Although this case is possible in principle, it is never observed in equilibrium because it is not optimal for banks to hoard sufficient liquidity for this case to arise. Comparing the demand for liquidity with the supply of liquidity in case of the liquidity shock, it is clear that the fire sales arises if and only if  $\lambda cn$  is greater than  $bn$ . In other words fire sales are observed in equilibrium as long as  $\tilde{c} > b$ . Given that  $\tilde{c}$  is a parameter, the ex-ante liquidity choice of banks determine if fire sales occur. As we know from the benchmark case banks optimally set  $b_i < c$ . Because this is true for any parameter value, it is true for  $\tilde{c} < c$  as well. The intuition is the same simple one: Holding liquidity is very costly if the shock does not materialize. Thus for banks to hoard liquidity there must be some additional return to holding liquidity in case of the liquidity shock. This additional return is only possible if the fire sale price is less than  $R$ , which is then only possible if there are fire sales. In other words, if there will not be any fire sales in the bad state, that is, if  $P = R$ , then there is no benefit of holding liquidity. But this contradicts with the assumption of sufficient liquidity in the banking system.

## 6 Conclusion

In this paper, we investigate the optimal design of bank regulation when financial markets are incomplete and characterized by fire sale externalities. In the model, banks need to fire sell their long term assets if aggregate liquidity shocks hit the economy. In case of liquidity shocks, each bank determines the optimal amount of assets to sell, without taking into account the effect of its asset sales on price. This creates a system wide externality under which there is, a simultaneous over-investment in risky assets and under-provision of liquid assets in the competitive equilibrium. This fact creates the need for bank regulation in our model.

First, we study the introduction of a capital requirement alone. The capital requirement limits the risky investment levels of banks. *Ceteris paribus*, more stringent capital requirements lead to less severe financial crises. However, we show that banks respond by decreasing their liquidity

ratios—which, in turn, creates an opposing channel. The regulator, anticipating this response, sets capital ratios at even higher levels to offset the decrease in banks' liquidity ratios. However, higher capital ratios are not sufficient to protect the financial system against fire sales. Without enough liquidity buffers banks' capital can easily erode with the losses due to fire sales. Thus, under liquidity shocks, a well-capitalized banking system may experience greater losses than a less capitalized banking system with strong liquidity buffers.

Due to the inefficiently low liquidity ratios, seemingly high capital ratios will rapidly deteriorate in case of a liquidity shock. In other words, capital adequacy ratios are neither a robust measure of bank soundness nor a sufficient tool to regulate the excessive risk taking behavior of banks when fire sale externalities are present. We show that the optimal bank regulation should supplement capital adequacy ratios with liquidity requirements. In that regard, our results support the Basel III approach, which strengthens earlier capital adequacy accords by adding liquidity requirements.

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# 7 Appendix

## 7.1 Figures

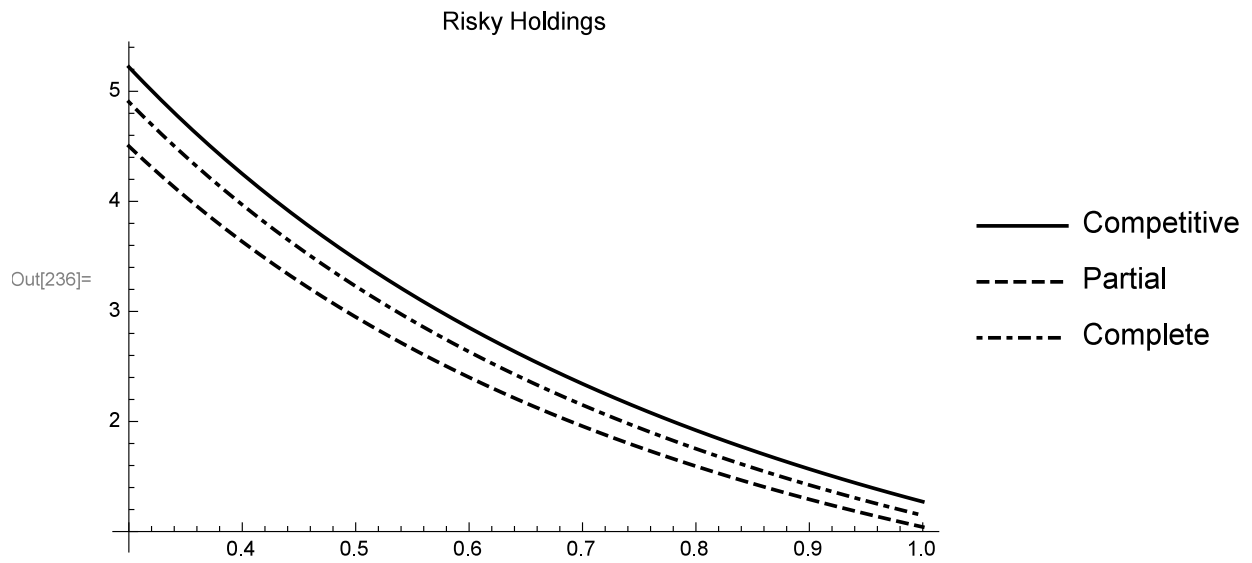


Figure 3: Risky asset holdings

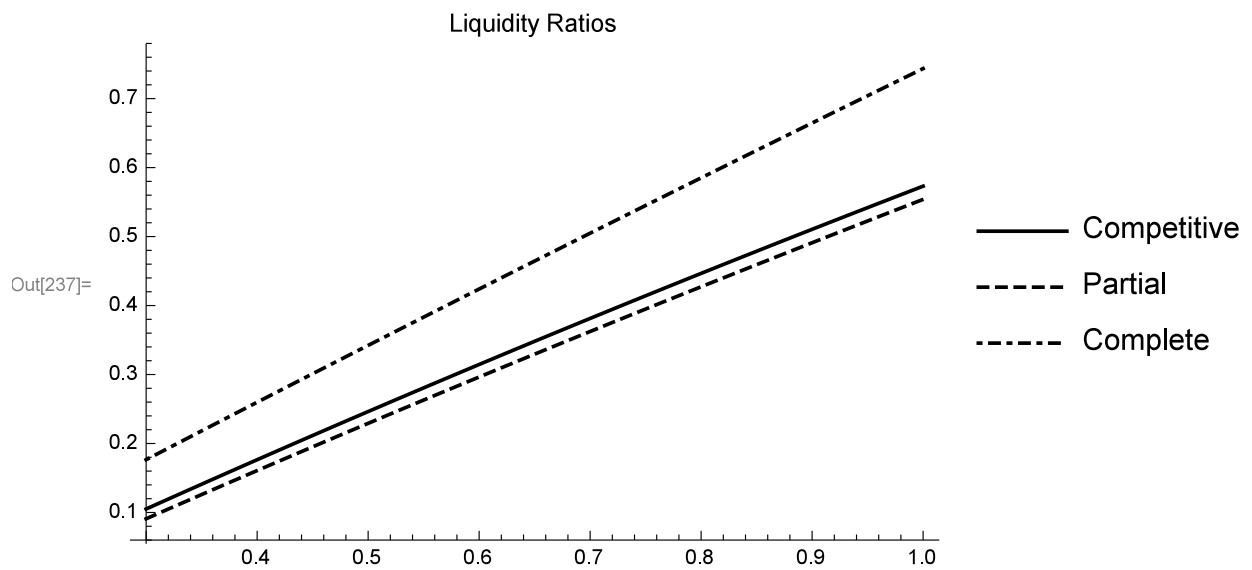


Figure 4: Liquidity ratios

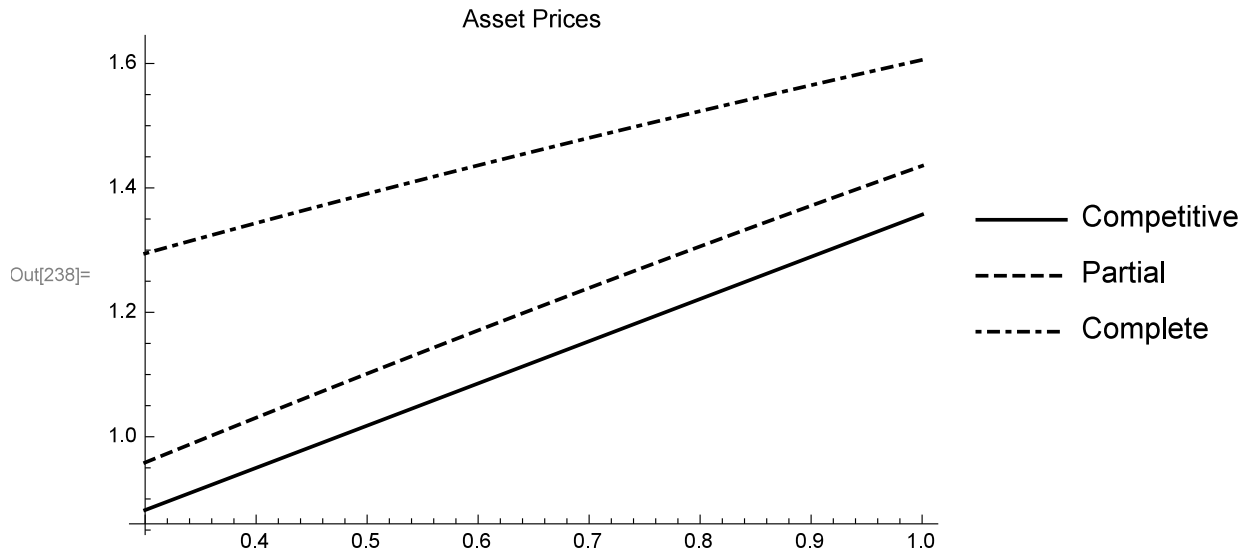


Figure 5: Price of the risky assets under fire sales

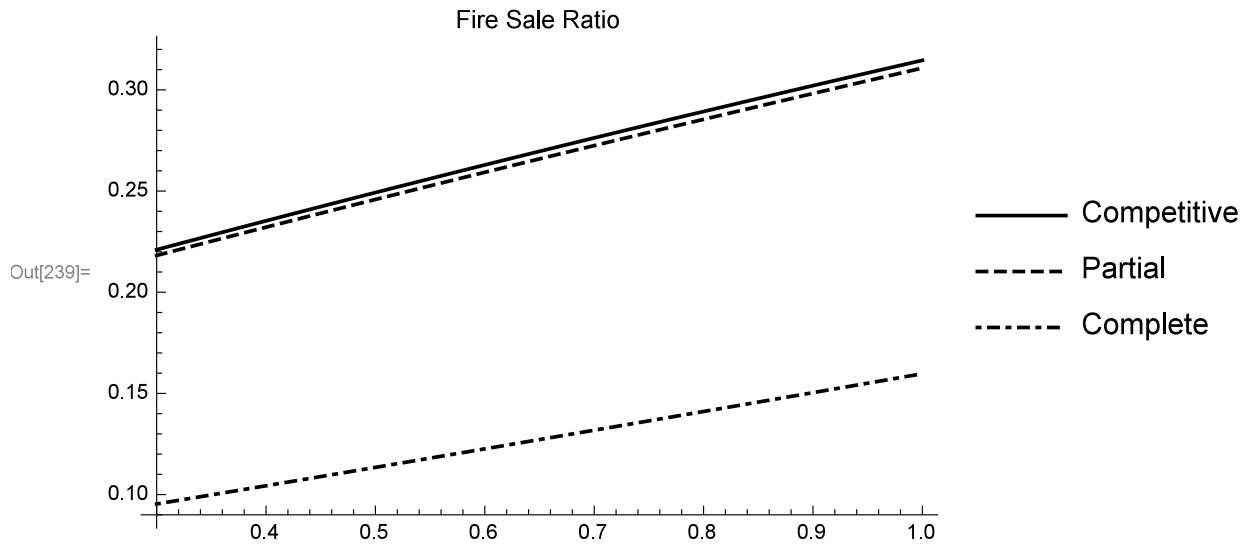


Figure 6: Fraction of assets sold in fire sales

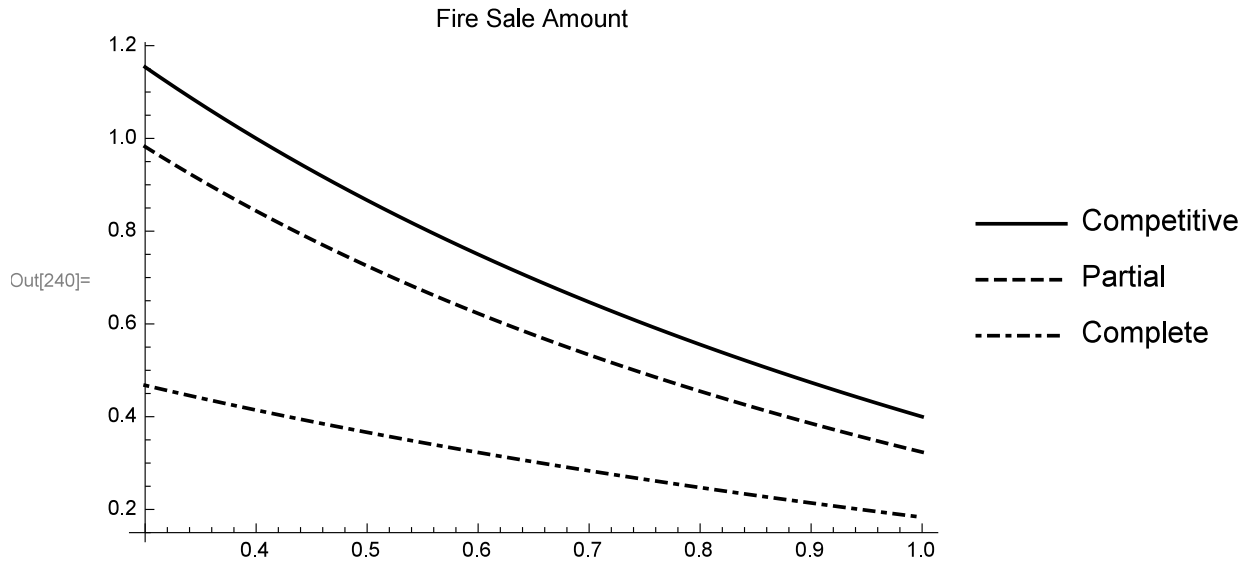


Figure 7: Total amount of fire sales

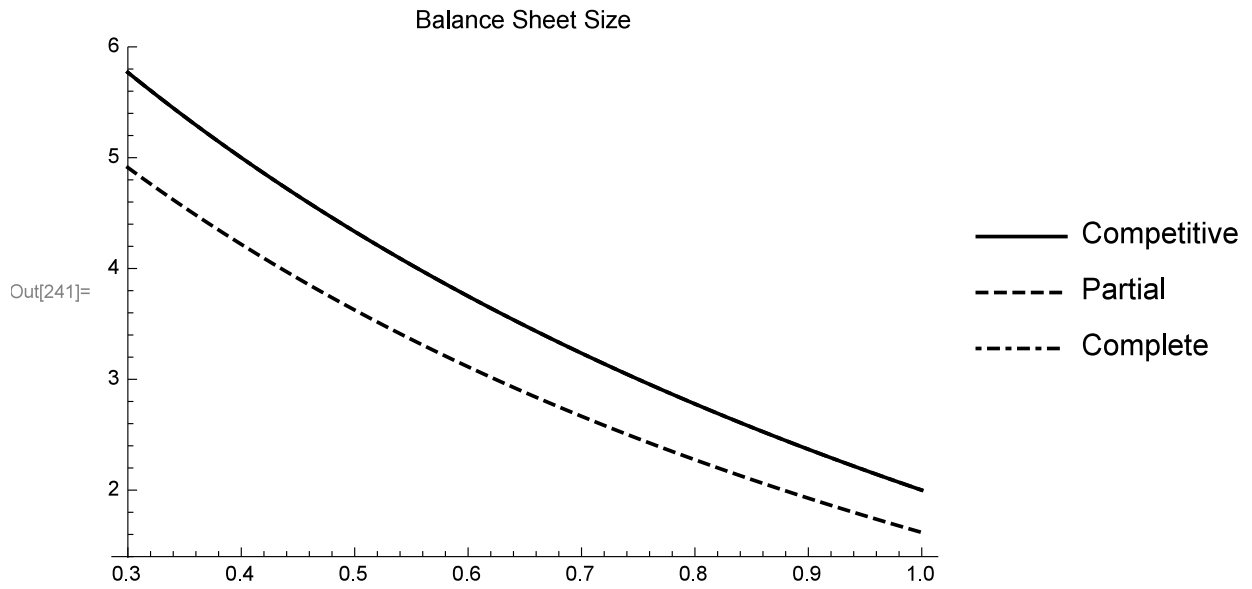


Figure 8: Balance Sheet Size

## 7.2 Closed–form solutions for the competitive equilibrium

Let global investors' technology function be given by  $F = Rln(1 + y)$ . Global investors choose how much assets,  $y$ , to buy from banks in the bad state at  $t = 1$  to maximize their profits,  $F(y) - Py$ , where  $P$  is the price of assets. The first order conditions of this problem yields the (inverse) demand function of global investors for risky assets:

$$P = F'(y) = \frac{R}{1 + y} \text{ and hence } y = F'^{-1}(P) = \frac{R}{P} - 1 \equiv Q^d(P). \quad (36)$$

We have solved for the competitive equilibrium price,  $P$ , in the main text, as shown by 10. Now, use this solution in the demand side function and define the total amount assets purchased by outside investors,  $\tau$ , in terms of the exogenous variables as follows:

$$y = \frac{R}{P} - 1 = \frac{R - 1 + q}{q(1 + c)} - 1 \equiv \tau. \quad (37)$$

On the other hand, the total supply of asset by banks was obtained as  $(1 - \gamma)n$  by (5). Hence, the market clearing condition,  $(1 - \gamma)n = \tau$ , yields:

$$(c - b)n = P\tau \implies n = \frac{P\tau}{c - b} \quad (38)$$

This equation gives the investment level,  $n$ , as a function of the liquidity ratio,  $b$ . We can solve for the latter from the first order conditions of banks' problem in the decentralized case, given by (60). Using  $\frac{R}{P} = \tau + 1$  from (12) and the functional form of the operational cost,  $\Phi'(n(1 + b)) = 2dn(1 + b)$ , in the first order conditions yields:

$$\begin{aligned} 1 - q + q(\tau + 1) &= 1 + 2dn(1 + b) \\ 1 + q\tau &= 1 + 2d\frac{P\tau}{c - b}(1 + b), \end{aligned}$$

where in the second line we used  $n = P\tau/(c - b)$  from (38). Finally substitute  $P = R/(\tau + 1)$  from (37) and simplify to obtain the liquidity ratio in the competitive equilibrium:

$$b = \frac{cq(\tau + 1) - 2dR}{q(\tau + 1) + 2dR}.$$

Investment level in the competitive equilibrium can as be obtained by using the solution for  $b$  in (37).

## 7.3 Closed–form solutions for the partial regulation case

The regulator takes into account that for any given  $n$ , the banks optimally choose their liquidity ratio  $b(n)$ , and hence we can write the regulator's objective function as:

$$\max_n W(n) = (1 - q)\{R + b(n)\}n + qR\gamma n - D((1 + b(n))n),$$

from which we can obtain the following first order conditions with respect to  $n$  as

$$(1 - q)\{R + b(n) + nb'(n)\} + qR \left\{ \gamma + n \frac{d\gamma}{dn} \right\} = D'(n(1 + b))\{1 + b(n) + nb'(n)\}. \quad (39)$$

First, note that by (4) we have that

$$\gamma = 1 + \frac{b(n) - c}{P} = 1 + \frac{b(n) - c}{R + (b(n) - c)n},$$

Using this, we can obtain the total derivative of  $\gamma$  with respect to  $n$  as:

$$\begin{aligned} \frac{d\gamma}{dn} &= \frac{\partial\gamma}{\partial b}b'(n) + \frac{\partial\gamma}{\partial n} \\ &= \frac{P - (b(n) - c)n}{P^2}b'(n) - \frac{(b(n) - c)^2}{P^2} \\ &= \frac{b'(n)}{P} - \frac{nb'(n)(b(n) - c)}{P^2} - \frac{(b(n) - c)^2}{P^2} \end{aligned} \quad (40)$$

Substitute this into the first order conditions given by (39) and rearrange

$$\begin{aligned} (1 - q)\{R + b(n)\} + qR \left( 1 + \frac{b(n) - c}{P} \right) + nb'(n) \left\{ 1 - q + \frac{qR}{P} - D'(\cdot) - \frac{(b(n) - c)n}{P^2}qR \right\} \\ - qR \frac{n(b(n) - c)^2}{P^2} - D'(\cdot)\{1 + b(n)\} = 0 \end{aligned}$$

Note that from banks' first order condition we have  $1 - q + qR/P - D'(\cdot) = 0$ , hence the equation above further simplifies to

$$\begin{aligned} R - qR + (1 - q)b(n) + qR + qR \frac{b(n) - c}{P} - qR \frac{n(b(n) - c)^2}{P^2} - b'(n) \left\{ \frac{(b(n) - c)n^2}{P^2}qR \right\} \\ - \left( 1 - q + \frac{qR}{P} \right) \{1 + b(n)\} = 0 \\ R - 1 + q + \frac{qR}{P}[b(n) - c - 1 - b(n)] - qR \frac{n(b(n) - c)^2}{P^2} - b'(n) \left\{ \frac{(b(n) - c)n^2}{P^2}qR \right\} = 0 \\ R - 1 + q - \frac{qR(1 + c)}{P} - qR \frac{n(b(n) - c)^2}{P^2} - b'(n) \left\{ \frac{(b(n) - c)n^2}{P^2}qR \right\} = 0 \\ R - 1 + q - \frac{qR(1 + c)[R + (b(n) - c)n] - qR(b(n) - c)^2n}{P^2} - b'(n) \left\{ \frac{(b(n) - c)n^2}{P^2}qR \right\} = 0 \\ R - 1 + q - \frac{qR^2(1 + c)}{P^2} - qR \frac{n(b(n) - c)(1 + b(n))}{P^2} - b'(n) \left\{ \frac{(b(n) - c)n^2}{P^2}qR \right\} = 0 \end{aligned} \quad (41)$$

Divide the last equation by  $qR$  to obtain

$$\frac{R - 1 + q}{qR} - \frac{R(1 + c)}{P^2} - \frac{n(b(n) - c)(1 + b(n))}{P^2} - b'(n) \left\{ \frac{(b(n) - c)n^2}{P^2} \right\} \quad (42)$$

Let's define

$$\sigma = \frac{R - 1 + q}{qR} \quad (43)$$

Now, we can write the first order condition as

$$\frac{1}{P^2} \left\{ \sigma P^2 - R(1 + c) - n(b(n) - c)(1 + b(n)) - b'(n)(b(n) - c)n^2 \right\} = 0 \quad (44)$$

We focus on the term inside the curly brackets because in equilibrium price must be strictly positive. Use the expressions obtained for  $1 + b$ ,  $b - c$  and  $b'(n)$  in the proof of Proposition 4, to rewrite this term as:

$$\begin{aligned} \sigma P^2 - R(1 + c) + (R - P) \frac{q(R - P)}{2dPn} - \frac{q(R - P)^2}{n^2[2dP^2 + qR]} \frac{R - P}{n} n^2 &= 0 \\ \sigma P^2 - R(1 + c) + \frac{q(R - P)^2}{n} \left[ \frac{1}{2dP} + \frac{R - P}{2dP^2 + qR} \right] &= 0 \end{aligned}$$

From the last equation we can obtain  $n$  in terms of  $P$  and the parameters of the model:

$$n = \frac{q(R - P)^2 \left[ \frac{1}{2dP} + \frac{R - P}{2dP^2 + qR} \right]}{R(1 + c) - \sigma P^2} \equiv \psi(P) \quad (45)$$

We can similarly obtain an expression for  $b$  in terms of  $P$  and the parameters of the model using the equilibrium price function  $P = R + (b - c)n$ , which implies that

$$b = \frac{P - R}{n} + c = \frac{P - R + cn}{n} = \frac{P - R + c\psi(P)}{\psi(P)} \quad (46)$$

Now, go back to the banks' first order condition, substitute these expressions for  $n$  and  $b$  in order to obtain a fixed point equation that involves only  $P$  as an endogenous variable, from which we can solve for the equilibrium price  $P$ .

$$\begin{aligned} -q + \frac{qR}{P} &= 2dn(1 + b) \\ \frac{qR}{P} &= 2d\psi(P) \left[ \frac{P - R + c\psi(P)}{\psi(P)} + 1 \right] + q \\ \frac{qR}{P} &= 2d\psi(P) \left[ \frac{P - R + (1 + c)\psi(P)}{\psi(P)} \right] + q \end{aligned}$$

Multiply the last equation with  $P$  and rearrange to obtain

$$\begin{aligned} 2d[P - R + (1 + c)\psi(P)] + qP - qR &= 0 \\ -2dP(R - P) - q(R - P) + 2d(1 + c)P\psi(P) &= 0 \end{aligned}$$

Rearrange the last equation and substitute for  $\psi(P)$  from (45):

$$\begin{aligned}
2d(1+c)P\psi(P) &= (R-P)(2dP+q) \\
2d(1+c)P \frac{q(R-P)^2 \left[ \frac{1}{2dP} + \frac{R-P}{2dP^2+qR} \right]}{R(1+c) - \sigma P^2} &= (R-P)(2dP+q) \\
2d(1+c)qP(R-P) \left[ \frac{1}{2dP} + \frac{R-P}{2dP^2+qR} \right] &= [R(1+c) - \sigma P^2] (2dP+q) \\
(1+c)q(R-P) \left[ 1 + \frac{2dP(R-P)}{2dP^2+qR} \right] &= [R(1+c) - \sigma P^2] (2dP+q) \\
(1+c)q(R-P) \left[ \frac{2dP^2+qR+2dPR-2dP^2}{2dP^2+qR} \right] &= [R(1+c) - \sigma P^2] (2dP+q) \\
(1+c)q(R-P) \left[ \frac{R(2dP+q)}{2dP^2+qR} \right] &= [R(1+c) - \sigma P^2] (2dP+q)
\end{aligned}$$

Lastly, simplifying  $2dP+q$  from both sides and rearranging yields

$$(1+c)q(R-P)R - (2dP^2+qR) [R(1+c) - \sigma P^2] = 0 \quad (47)$$

Rewrite this equation first by substituting  $\beta \equiv R(1+c)$ , and then expand it to obtain a polynomial equation in  $P$ :

$$\begin{aligned}
q(R-P)\beta - (2dP^2+qR) [\beta - \sigma P^2] &= 0 \\
qR\beta - q\beta P - 2d\beta P^2 + 2d\sigma P^4 - qR\beta + qR\sigma P^2 &= 0 \\
2d\sigma P^4 + (qR\sigma - 2d\beta)P^2 - q\beta P &= 0
\end{aligned}$$

Because we are interested in non-zero and positive equilibrium price for the illiquid asset, divide this last equation by  $P$  to obtain a cubic equation in  $P$ :

$$2d\sigma P^3 + [qR\sigma - 2d\beta]P - q\beta = 0 \quad (48)$$

It is easy to show that this cubic equation has only one real root and two complex conjugate roots. The only real root can easily be obtained using Vieta's substitution for cubic equations. Because this real root gives the equilibrium price  $P$  in terms of the model parameters, we can also obtain a closed-form solution for the equilibrium level of risky investment  $n$  by substituting this value of  $P$  into (45), and for the equilibrium value of liquid asset ratio  $b$  by substituting the solution for  $P$  into (46).

#### 7.4 Closed-form solutions for the complete regulation case

The first order conditions of the regulator's problem in the complete regulation case, given by (20), with respect to  $n$  and  $b$ , are respectively:

$$\begin{aligned}
(1-q)(R+b) + qR \left\{ \gamma + \frac{\partial \gamma}{\partial n} n \right\} &= D'(n(1+b))(1+b), \\
(1-q)n + qR \frac{\partial \gamma}{\partial b} n &= D'(n(1+b))n,
\end{aligned} \quad (49)$$

where  $\gamma = 1 + \frac{b-c}{P}$ . Combine the two equations to obtain:

$$(1-q)(R+b) + qR\left\{\gamma + \frac{\partial\gamma}{\partial n}n\right\} = \left[(1-q) + qR\frac{\partial\gamma}{\partial b}\right](1+b) = D'(n(1+b))(1+b)$$

Substitute using  $\frac{\partial\gamma}{\partial n} = -\frac{(b-c)^2}{P^2}$  and  $\frac{\partial\gamma}{\partial b} = \frac{R}{P^2}$ , and later  $P = R + (b-c)n$ .

$$\begin{aligned} -\phi &= -\frac{(1-q)[R-1]}{qR} = 1 + \frac{(b-c)P - (b-c)^2n - R(1+b)}{P^2} \\ -\phi - 1 &= \frac{(b-c)[R + (b-c)n] - (b-c)^2n - R(1+b)}{P^2} = \frac{R(b-c-1-b)}{P^2} = \frac{-R(c+1)}{P^2} \\ \implies P^2 &= \frac{R(c+1)}{\phi+1} = \frac{q(c+1)R^2}{R-1+q} \implies P^{**} = R\sqrt{\frac{q(c+1)}{R-1+q}} \end{aligned} \tag{50}$$

Note that there is simple relationship between the competitive equilibrium price and the price under complete regulation, captured by  $P = P^{**2}/R$ .

To solve for the equilibrium outcomes, go back to the first order condition with respect to  $b$  given above:

$$\begin{aligned} (1-q) + qR\frac{\partial\gamma}{\partial b} &= D'(n(1+b)) \\ 1-q + qR\frac{R}{P^2} &= 1 + 2dn(1+b) \\ 1-q + q\left(\frac{R}{P}\right)^2 &= 1 + 2dn(1+b) \\ 1-q + q(\tau^{**} + 1)^2 &= 1 + 2dn(1+b) \\ q\{(\tau^{**} + 1)^2 - 1\} &= 2dn(1+b) \\ q\{(\tau^{**} + 1 + 1)(\tau^{**} + 1 - 1)\} &= 2d\frac{P\tau^{**}}{c-b}(1+b) \\ q\tau^{**}(\tau^{**} + 2) &= 2d\frac{R}{\tau^{**} + 1}\frac{\tau^{**}}{c-b}(1+b) \\ q(\tau^{**} + 1)(\tau^{**} + 2)(c-b) &= 2dR(1+b) \\ q(\tau^{**} + 1)(\tau^{**} + 2)c - 2dR &= b\{2dR + q(\tau^{**} + 1)(\tau^{**} + 2)\} \\ b^{**} &= \frac{cq(\tau^{**} + 1)(\tau^{**} + 2) - 2dR}{2dR + q(\tau^{**} + 1)(\tau^{**} + 2)} = \frac{cq - \frac{2dR}{(\tau^{**} + 1)(\tau^{**} + 2)}}{\frac{2dR}{(\tau^{**} + 1)(\tau^{**} + 2)} + q} \end{aligned}$$

where we used  $R/P^{**} = \tau^{**} + 1$  and  $n^{**} = P^{**}\tau^{**}/(c-b)$ .



## 7.5 Proofs omitted in the main text

**Lemma 1.**  $P(n, b)$  is decreasing in  $n$  and increasing in  $b$ .

*Proof.* The asset market clearing condition in the bad state at  $t = 1$  is given as

$$Q^s(P) = \frac{c - b}{P}n = Q^d(P), \quad (51)$$

which can be written as

$$(c - b)n = PQ^d(P). \quad (52)$$

First, take the partial derivative of both sides of this last equation with respect to  $n$ :

$$\begin{aligned} c - b &= \frac{\partial P}{\partial n}Q^d(P) + P\frac{\partial Q^d(P)}{\partial P}\frac{\partial P}{\partial n} \\ &= \frac{\partial P}{\partial n}\left\{Q^d(P) + P\frac{\partial Q^d(P)}{\partial P}\right\} \\ &= \frac{\partial P}{\partial n}Q^d(P)\{1 + \epsilon^d\} \end{aligned}$$

where

$$\epsilon^d = \frac{\partial Q^d(P)}{\partial P}\frac{P}{Q^d}, \quad (53)$$

is the price elasticity of demand function of the outside investors. Rearranging the last equation gives

$$\frac{\partial P}{\partial n} = \frac{c - b}{Q^d(P)(1 + \epsilon^d)} < 0 \quad (54)$$

since  $1 + \epsilon^d < 0$  by Assumption *Elasticity*, and  $c - b > 0$  by assumption here because we are examining the case with fire sales. We will later show in Lemma 3 that  $c - b > 0$  actually holds in equilibrium.

For the second part of the proof take the partial derivative of both sides of (52) with respect to  $b$ :

$$\begin{aligned} -n &= \frac{\partial P}{\partial b}Q^d(P) + P\frac{\partial Q^d(P)}{\partial P}\frac{\partial P}{\partial b} \\ &= \frac{\partial P}{\partial b}\left\{Q^d(P) + P\frac{\partial Q^d(P)}{\partial P}\right\} \\ &= \frac{\partial P}{\partial b}Q^d(P)\{1 + \epsilon^d\} \end{aligned}$$

Rearranging the last equation gives

$$\frac{\partial P}{\partial b} = -\frac{n}{Q^d(P)(1 + \epsilon^d)} > 0 \quad (55)$$

since  $1 + \epsilon^d < 0$  by Assumption *Elasticity*.

□

**Lemma 2.** *Equilibrium fraction of assets sold,  $1 - \gamma(n, b)$ , is increasing in  $n$  and decreasing in  $b$ .*

*Proof.* Using (4) we can write banks' asset sales in equilibrium as  $1 - \gamma(n, b) = (c - b)/P(n, b)$ . Note that

$$\frac{\partial(1 - \gamma)}{\partial n} = \frac{\partial(1 - \gamma)}{\partial P} \frac{\partial P}{\partial n} > 0, \quad (56)$$

because  $\partial(1 - \gamma)/\partial P = -c/P^2 < 0$  from (4) and by Lemma 1 we have that  $\partial P/\partial n < 0$ . Similarly, we can obtain

$$\frac{\partial(1 - \gamma)}{\partial b} = -\frac{1}{P} + \frac{\partial(1 - \gamma)}{\partial P} \frac{\partial P}{\partial b} < 0, \quad (57)$$

since  $\partial(1 - \gamma)/\partial P < 0$  as shown above, and by Lemma 1 we have that  $\partial P/\partial b > 0$ .  $\square$

**Proposition 2.** *The competitive equilibrium price of assets is given by:*

$$P = \frac{qR(1 + c)}{R - 1 + q}. \quad (58)$$

*The equilibrium price,  $P$ , is increasing in the probability of the liquidity shock,  $q$ , and the size of the shock,  $c$ , but decreasing in the return on the risky assets,  $R$ .*

*Proof.* The first order conditions of the banks' problem (7) with respect to  $n_i$  and  $b_i$  in this case respectively are:

$$(1 - q)(R + b_i) + qR\gamma_i = D'(n_i(1 + b_i))(1 + b_i), \quad (59)$$

$$(1 - q)n_i + qR\frac{1}{P}n_i = D'(n_i(1 + b_i))n_i, \quad (60)$$

where  $\gamma_i = 1 - (c - b_i)/P$  as obtained in the previous section. Combining the two equations gives:

$$(1 - q)R + (1 - q)b_i + qR + qR\left(\frac{b_i - c}{P}\right) = (1 - q) + (1 - q)b_i + \frac{qR}{P} + \frac{qR}{P}b_i.$$

In this last equation, the terms that involve the liquidity ratio,  $b_i$ , on both sides cancel out each other, and hence we can solve for  $P$ , the competitive equilibrium price of assets. It is straightforward to obtain the sign of the derivative of the equilibrium price with respect to model parameters,  $R, c, q$ .  $\square$

**Proposition 3.** *The comparative statics for the competitive equilibrium risky investment level,  $n$ , and liquidity ratio,  $b$ , are as follows:*

1. *The risky holdings in the competitive equilibrium ( $n$ ) are increasing in the return to the risky asset ( $R$ ), and decreasing in the size of the liquidity shock ( $c$ ), marginal cost of funds ( $d$ ), and the probability of the bad state ( $q$ ).*
2. *The liquidity ratio in the competitive equilibrium ( $b$ ) is increasing in the size of the liquidity shock ( $c$ ), the return to the risky asset ( $R$ ) and the probability of the bad state ( $q$ ), and decreasing in the marginal cost of funds ( $d$ ).*

*Proof. Part 1:*

$$\begin{aligned}
\frac{\partial b}{\partial c} &= \frac{\{q - 2dR(\tau + 1)^{-2}(-1)\frac{\partial \tau}{\partial c}\}[2d\frac{R}{\tau+1} + q] - [cq - 2d\frac{R}{\tau+1}]2dR(\tau + 1)^{-2}(-1)\frac{\partial \tau}{\partial c}}{[2d\frac{R}{\tau+1} + q]^2} \\
&= \frac{2dR(\tau + 1)^{-2}(-1)\frac{\partial \tau}{\partial c}[-2d\frac{R}{\tau+1} - q - cq + 2d\frac{R}{\tau+1}] + q[2d\frac{R}{\tau+1} + q]}{[2d\frac{R}{\tau+1} + q]^2} \\
&= \frac{2dR(\tau + 1)^{-2}(-1)\frac{\partial \tau}{\partial c}(-q)(1 + c) + q[2d\frac{R}{\tau+1} + q]}{[2d\frac{R}{\tau+1} + q]^2} \\
&= \frac{2dR(\tau + 1)^{-2}\frac{\partial \tau}{\partial c}q(1 + c) + q[2d\frac{R}{\tau+1} + q]}{[2d\frac{R}{\tau+1} + q]^2} \\
&= \frac{2dR(\frac{q(1+c)}{R-1+q})^2\frac{R-1+q}{q(1+c)^2}(-1)q(1 + c) + q[2dR\frac{q(1+c)}{R-1+q} + q]}{[2d\frac{R}{\tau+1} + q]^2} \\
&= \frac{2dR\frac{q(1+c)}{R-1+q}(-1)q + q[2dR\frac{q(1+c)}{R-1+q} + q]}{[2d\frac{R}{\tau+1} + q]^2} \\
&= \frac{q^2}{[2d\frac{R}{\tau+1} + q]^2} > 0.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial b}{\partial d} &= \frac{[-2\frac{R}{\tau+1}][2d\frac{R}{\tau+1} + q] - [cq - 2d\frac{R}{\tau+1}][2\frac{R}{\tau+1}]}{[2d\frac{R}{\tau+1} + q]^2} \\
&= \frac{[-2\frac{R}{\tau+1}][2d\frac{R}{\tau+1}] - [2\frac{R}{\tau+1}]q - [2\frac{R}{\tau+1}]cq + [2\frac{R}{\tau+1}][2d\frac{R}{\tau+1}]}{[2d\frac{R}{\tau+1} + q]^2} \\
&= \frac{-\frac{2R}{\tau+1}q(1 + c)}{[2d\frac{R}{\tau+1} + q]^2} < 0.
\end{aligned}$$

$$\frac{\partial b}{\partial q} = \frac{2(1 + c)^2dR}{(-1 + q + R + 2dR + 2cdR)^2} > 0$$

$$\begin{aligned}
\frac{\partial b}{\partial R} &= \frac{\{-2d \frac{(\tau+1)-R \frac{\partial(\tau+1)}{\partial R}}{(\tau+1)^2}\} [2d \frac{R}{\tau+1} + q] - [cq - 2d \frac{R}{\tau+1}] \{2d \frac{(\tau+1)-R \frac{\partial(\tau+1)}{\partial R}}{(\tau+1)^2}\}}{[2d \frac{R}{\tau+1} + q]^2} \\
&= \frac{\{-2d \frac{\frac{R-1+q}{q(1+c)} - \frac{R}{q(1+c)}}{(\tau+1)^2}\} [2d \frac{R}{\tau+1} + q] - [cq - 2d \frac{R}{\tau+1}] \{2d \frac{\frac{R-1+q}{q(1+c)} - \frac{R}{q(1+c)}}{(\tau+1)^2}\}}{[2d \frac{R}{\tau+1} + q]^2} \\
&= \frac{\{\frac{2d(1-q)}{q(1+c)(\tau+1)^2}\} [2d \frac{R}{\tau+1} + q] - [cq - 2d \frac{R}{\tau+1}] \{\frac{-2d(1-q)}{q(1+c)(\tau+1)^2}\}}{[2d \frac{R}{\tau+1} + q]^2} \\
&= \frac{\{\frac{2d(1-q)}{q(1+c)(\tau+1)^2}\} 2d \frac{R}{\tau+1} + \{\frac{2d(1-q)}{q(1+c)(\tau+1)^2}\} q + \{\frac{2d(1-q)}{q(1+c)(\tau+1)^2}\} cq - \{\frac{2d(1-q)}{q(1+c)(\tau+1)^2}\} 2d \frac{R}{\tau+1}}{[2d \frac{R}{\tau+1} + q]^2} \\
&= \frac{\{\frac{2d(1-q)}{q(1+c)(\tau+1)^2}\} q(1+c)}{[2d \frac{R}{\tau+1} + q]^2} \\
&= \frac{\frac{2d(1-q)}{(\tau+1)^2}}{[2d \frac{R}{\tau+1} + q]^2} > 0
\end{aligned}$$

**Part 2:**

$$\frac{\partial n}{\partial c} = \frac{(-1+c)q - 2(-1+(1+d+cd)R)}{2(1+c)^3 d} < 0, \text{ not obvious but clear after little algebra}$$

$$\frac{\partial n}{\partial d} = \frac{1+cq-R}{2(1+c)^2 d^2} < 0 \text{ due to assumption } R - cq > 1$$

$$\frac{\partial n}{\partial R} = \frac{(-1+q+R)^2 - 2(1+c)d(cq^2 - (-1+R)^2 - q(-1+c+2R))}{2(1+c)^2 d(-1+q+R)^2} > 0$$

$$\begin{aligned}
\underbrace{(-1+q+R)^2}_{\text{positive}} - 2(1+c)d(cq^2 - (-1+R)^2 - q(-1+c+2R)) &= . \\
&= .
\end{aligned}$$

$$(-1+q+R)^2 - 2(1+c)d(\underbrace{cq(q-1)}_{\text{negative}} - (-1+R)^2 - \underbrace{q(-1+2R)}_{\text{positive}}) > 0$$

$$\frac{\partial n}{\partial q} = -\frac{2d(-1+R)R + 2c^2 d(-1+R)R + c(q^2 + 2q(-1+R) + (-1+R)(-1+R+4dR))}{2(1+c)^2 d(-1+q+R)^2} < 0$$

□

**Proposition 4.** *Let the operational cost of a bank be given by  $\Phi(x) = dx^2$  and the [Technology assumption](#) holds for banks' return on the risky investment. Then for any technology for outside*

investors  $F$  that satisfies the *Concavity*, *Elasticity* and *Regularity* assumptions with  $F'(0) = R$ , banks decrease their liquidity ratio as the regulator tightens capital requirements, that is,  $b'(n) \geq 0$ .

*Proof.* We are studying the case of partial regulation, in which banks are free to choose their liquidity ratio  $b_i$  but the regulator limits their choice of  $n_i$ . Therefore we can modify the banks' profit as follows:

$$\Pi(n, b_i) = (1 - q)\{R + b_i\}n + qR\gamma_i - D((1 + b_i)n)$$

Here we can treat  $n$  like a parameter. The regulator in a sense determines the aggregate amount of  $n$ . Therefore, the first order conditions of the banks' problem above is

$$\begin{aligned} \frac{\partial \Pi(n, b_i)}{\partial b_i} &= (1 - q)n + qRn \frac{\partial \gamma}{\partial b_i} - n - 2dn^2(1 + b_i) = 0 \\ &= (1 - q)n + qRn \frac{1}{P} - n - 2dn^2(1 + b_i) = 0, \end{aligned}$$

which can be simplified as

$$q\left(\frac{R}{P} - 1\right) = 2dn(1 + b_i) \tag{61}$$

Note that we can obtain the derivative of equilibrium price with respect to the regulatory parameter,  $n$ , as follows:

$$\frac{\partial P}{\partial n} = \frac{F_2(c - b_i)}{F_1 + QF_2}, \tag{62}$$

where  $F_i \equiv \frac{d^i F(Q)}{dQ^i}$ .

Banks' profit function exhibits increasing differences in  $b_i$  and  $n$  if the cross derivative is positive. Increasing differences mean that  $b'(n) > 0$ , that is, the optimal choice of  $b_i$  in banks' problem is increasing with the regulatory parameter. Now using this, we can obtain the cross derivative of

banks' profit as:

$$\begin{aligned}
\frac{\partial^2 \Pi(n, b_i)}{\partial b_i \partial n} &= (1 - q) + qR \left( \frac{1}{P} - \frac{n}{P^2} \frac{\partial P}{\partial n} \right) - 1 - 4dn(1 + b_i) \\
&= (1 - q) + qR \left( \frac{1}{P} - \frac{n}{P^2} \frac{F_2(c - b_i)}{F_1 + QF_2} \right) - 1 - 2q \left( \frac{R}{P} - 1 \right) \\
&= -q + qR \left( \frac{1}{P} - \frac{n(c - b_i)}{PQ} \frac{1}{P} \frac{QF_2}{F_1 + QF_2} \right) - 2q \left( \frac{R}{P} - 1 \right) \\
&= -q + qR \left( \frac{1}{P} - \frac{n(c - b_i)}{PQ} \frac{1}{P} \frac{QF_2}{F_1 + QF_2} \right) - 2q \frac{R}{P} + 2q \\
&= -1 + R \left( \frac{1}{P} - \frac{n(c - b_i)}{PQ} \frac{1}{P} \frac{QF_2}{F_1 + QF_2} \right) - 2 \frac{R}{P} + 2 \\
&= 1 + R \left( \frac{1}{P} - \frac{1}{P} \frac{QF_2}{F_1 + QF_2} \right) - 2 \frac{R}{P} \\
&= 1 + \frac{R}{P} - \frac{R}{P} \frac{QF_2}{F_1 + QF_2} - 2 \frac{R}{P} \\
&= 1 - R \left( \frac{1}{P} + \frac{1}{P} \frac{QF_2}{F_1 + QF_2} \right) \\
&= 1 - R \frac{[F_1 + 2QF_2]}{F_1(F_1 + QF_2)} \\
\implies R &< \frac{F_1(F_1 + QF_2)}{F_1 + 2QF_2} \equiv \kappa \quad \text{implies increasing differences}
\end{aligned}$$

Redefine some of the objects;  $\kappa \frac{F_1(F_1 + QF_2)}{F_1 + 2QF_2}$ ,  $\frac{dF(Q)}{dQ} \equiv F_1$ ,  $\frac{d^2 F(Q)}{dQ^2} \equiv F_2$  and  $\frac{d^3 F(Q)}{dQ^3} \equiv F_3$

$$\begin{aligned}
\frac{d\kappa}{dQ} &= \left( [F_2(F_1 + QF_2) + F_1(F_2 + F_2 + QF_3)][F_1 + 2QF_2] \right) / (F_1 + 2QF_2)^2 \\
&= \left( [3F_1F_2 + QF_2^2 + F_1F_3Q][F_1 + QF_2 + QF_2] - [F_1(F_1 + QF_2)][F_2 + 2F_2 + 2QF_3] \right) / (F_1 + 2QF_2)^2
\end{aligned}$$

Focus on the numerator to get the sign of the derivative

$$\begin{aligned}
\frac{d\kappa}{dQ} \times (F_1 + 2QF_2)^2 &= [3F_1F_2 + QF_2^2 + F_1F_3Q - 3F_1F_2 - 2QF_1F_3](F_1 + QF_2) + QF_2[3F_1F_2 + QF_2^2 + F_1F_3Q] \\
&= Q(F_2^2 - F_1F_3)(F_1 + QF_2) + QF_2[3F_1F_2 + QF_2^2 + F_1F_3Q] \\
&= Q(F_2^2 - F_1F_3)F_1 + QF_2[QF_2^2 - QF_1F_3 + 3F_1F_2 + QF_2^2 + QF_1F_3] \\
&= Q(F_2^2 - F_1F_3)F_1 + QF_2[3F_1F_2 + 2QF_2^2] \quad \text{divide by } Q \\
&= F_1F_2^2 - F_1^2F_3 + 3F_1F_2^2 + 2QF_2^3 = 4F_1F_2 - F_1^2F_3 + 2QF_2^3 \\
&= 2F_1F_2 - F_1^2F_3 + 2F_1F_2 + 2QF_2^3 \\
&= F_1(2F_2^2 - F_1F_3) + 2F_2^2(F_1 + QF_2) > 0
\end{aligned}$$

$2F_2^2 - F_1F_3$  is positive due to the regularity assumption, and  $F_1 + QF_2$  is positive due to the elasticity assumption. □

**Proposition 5.** *It is socially optimal to take fire sale risk, that is, the socially optimal liquidity ratio satisfies  $b < c$ .*

*Proof.* It is important to note that in principle it is possible to completely insure against the fire sale risk. In this case, similar to some interpretation of narrow banking (Freixas & Rochet(2008) 7.2.2), the banks are able to cover the liquidity need even in the worst case by using their liquidity holdings. However, we show that this is not optimal and the social planner takes some fire-sale risk, by setting the aggregate liquidity ratio less than the liquidity need  $c$ .

To show this we start with fully insured case and move  $\varepsilon$  amount of investment in risky asset to liquid asset, and show that this re-allocation is profitable. As a result of this re-allocation the banks are now exposed to fire sale risk, and the fire sale price is denoted as  $\bar{P}$ .

$$\begin{aligned}
E[\pi_i] &= (1 - q)(R + b)n + qR(1 - \frac{c - b}{P})n \\
&= (1 - q)Rn + (1 - q)B + qRn - qR\frac{cn}{P} + qR\frac{B}{P} \\
&= Rn + (1 - q)B - qR\frac{cn}{P} + qR\frac{B}{P} \\
\text{if } B = cn - \varepsilon &\implies R(n + \varepsilon) + (1 - q)(cn - \varepsilon) - qR\frac{c(n + \varepsilon)}{P} + qR\frac{cn - \varepsilon}{P} \\
&= Rn + (1 - q)B - qR\frac{cn}{P} + qR\frac{B}{P} + R\varepsilon - (1 - q)\varepsilon - qR\varepsilon\frac{1 + c}{P} \\
&= Rn + (1 - q)B - qR\frac{cn}{P} + qR\frac{B}{P} + \varepsilon(R - 1 + q - qR\frac{1 + c}{P}) \\
&> 0 \text{ if } \bar{P} > \frac{qR(1 + c)}{R - 1 + q}
\end{aligned}$$

As long as the fire sale price does not decrease dramatically, below the competitive equilibrium price to be specific, the moment just a small fraction of risky assets are sold to the outside investors, taking some fire sale risk is profitable. □

**Proposition 6.** *Risky investment levels, liquid asset holdings, and financial stability measures under competitive equilibrium  $(n, b, 1 - \gamma, P)$ , partial regulation equilibrium  $(n^*, b^*, 1 - \gamma^*, P^*)$ , and complete regulation equilibrium  $(n^{**}, b^{**}, 1 - \gamma^{**}, P^{**})$  compare as follows:*

a)  $n > n^{**} > n^*$

b)  $b^{**} > b > b^*$

c) *Financial stability measures*

i)  $1 - \gamma > 1 - \gamma^* > 1 - \gamma^{**}$

ii)  $(1 - \gamma)n > (1 - \gamma^*)n^* > (1 - \gamma^{**})n^{**}$

iii)  $P^{**} > P^* > P$

*Proof.* Proof of this proposition is established through a series of lemmas below. □

**Lemma 3.**  $P^{**} > P^* > P$

*Proof. Part 1:*  $P^{**} > P^*$ . Note that, in the solution to the complete regulation case above, we have obtained

$$P^{**} = \sqrt{\frac{qR^2(1+c)}{R-1+q}} = \sqrt{\frac{\beta}{\sigma}} \quad (63)$$

where  $\sigma$  is defined by (43) and  $\beta \equiv R(1+c)$ . Start from the cubic equation obtained in the solution for the partial case that gives  $P^*$ . We repeat this cubic equation below for convenience:

$$2d\sigma P^{*3} + [qR\sigma - 2d\beta]P^* - q\beta = 0 \quad (64)$$

Divide this equation by  $\sigma$  to obtain:

$$2dP^{*3} + \left[ qR - 2d\frac{\beta}{\sigma} \right] P^* - q\frac{\beta}{\sigma} = 0 \quad (65)$$

Note that  $\beta/\sigma = P^{**2}$ , and substitute this into the equation above and manipulate:

$$2dP^{*3} + [qR - 2dP^{**2}] P^* - qP^{**2} = 0 \quad (66)$$

$$2dP^{*3} + qRP^* - 2dP^{**2}P^* - qP^{**2} = 0 \quad (67)$$

$$(2dP^{*2} + qR)P^* = (2dP^{**2} + q)P^{**2} \quad (68)$$

Multiply both sides of this equation by  $P^*$  to obtain:

$$(2dP^{*2} + qR)P^{*2} = (2dP^{**2} + qP)P^{**2} \quad (69)$$

From this last equivalence we can obtain the square of the price ratios in the two cases as:

$$\left( \frac{P^{**}}{P^*} \right)^2 = \frac{2dP^{*2} + qR}{2dP^{**2} + qP} > 1, \quad (70)$$

since we must have  $R > P^*$  in equilibrium. Therefore,  $P^{**} > P^*$ .



**Part 2:**  $P^* > P$ . First, note that we obtained the competitive equilibrium price of assets in the main text as:

$$P = \frac{qR(1+c)}{R-1+q} = \frac{\beta}{R\sigma}, \quad (71)$$

using the definitions of  $\sigma, \beta$  as defined above. Now, take the cubic equation given by (64) and divide it  $R\sigma$  to obtain:

$$\frac{2d}{R}P^{*3} + \left[ q - 2d\frac{\beta}{R\sigma} \right] P^* - q\frac{\beta}{R\sigma} = 0 \quad (72)$$

Note that  $\beta/R\sigma = P$ , and substitute this into the equation above and manipulate:

$$\frac{2d}{R}P^{*3} + [q - 2dP] P^* - qP = 0 \quad (73)$$

$$\left( \frac{2d}{R}P^{*2} + q \right) P^* = (2dP^* + q)P \quad (74)$$

From this last equivalence we can obtain the price ratios in these two cases as:

$$\frac{P}{P^*} = \frac{\frac{2d}{R}P^{*2} + q}{2dP^* + q} = \frac{2dP^{*2} + qR}{2dRP^* + qR} < 1, \quad (75)$$

The last inequality holds because  $P^{*2} < RP^*$ , which is in turn true since we must have  $P^* < R$  in equilibrium. Therefore,  $P^* > P$ .  $\square$

**Lemma 4.**  $b^{**} > b > b^*$

*Proof. Part 1:*  $b^{**} > b$ . Note that the closed-form solutions for the liquidity ratios in these two cases were obtained above as:

$$b = \frac{cq - 2d\frac{R}{\tau+1}}{2d\frac{R}{\tau+1} + q} \quad (76)$$

$$b^{**} = \frac{cq - 2d\frac{R}{(\tau^{**}+1)(\tau^{**}+2)}}{2d\frac{R}{(\tau^{**}+1)(\tau^{**}+2)} + q} \quad (77)$$

where

$$\tau = \frac{R-1+q}{q(1+c)} - 1 = \eta^2 - 1 \quad (78)$$

$$\tau^{**} = \frac{R}{P^{**}} - 1 = \sqrt{\frac{R-1+q}{q(1+c)}} - 1 = \eta - 1 \quad (79)$$

Therefore,  $(\tau^{**} + 1)(\tau^{**} + 2) = \eta(\eta + 1) > \tau + 1 = \eta^2 \implies b^{**} > b$ .

**Part 2:**  $b > b^*$ . Start from first order conditions of banks' problem in the partial case:

$$\begin{aligned}
1 - q + \frac{qR}{P} &= 1 + 2dn(1 + b(n)) \\
-q + \frac{qR}{P} &= 2dn(1 + b(n)) \\
q\left(\frac{R}{P} - 1\right) &= 2d\frac{P\tau}{c - b(n)}(1 + b(n)) \\
q\tau &= 2d\frac{P\tau}{c - b(n)}(1 + b(n)) \\
\implies q(c - b(n)) &= 2dP(1 + b(n)) = 2d(1 + b(n))\frac{R}{\tau + 1} \\
\implies qc(\tau + 1) - qb(n)(\tau + 1) &= 2dR + 2db(n)R \implies qc(\tau + 1) - 2dR = b(n)\{2dR + q(\tau + 1)\} \\
\implies b(n) &= \frac{qc(\tau + 1) - 2dR}{2dR + q(\tau + 1)}
\end{aligned}$$

where we used  $n = \frac{P\tau}{c - b(n)}$  and  $P = \frac{R}{\tau + 1}$ .

The competitive equilibrium liquidity ratio ( $b$ ) and  $b(n)$  in the partial regulation case have the same functional form, the only difference is  $\tau$  vs  $\tau^*$ .

$$\frac{db}{d\tau} = \frac{qc[2dR + q(\tau + 1)] - q[qc(\tau + 1) - 2dR]}{[2dR + q(\tau + 1)]^2} = \frac{2dRq(1_c)}{[2dR + q(\tau + 1)]^2} > 0 \quad (80)$$

Therefore,  $b > b^*$  because  $\tau > \tau^*$ . □

**Lemma 5.**  $n > n^{**} > n^*$

*Proof. Part 1:*  $n > n^{**}$ . We will use  $n^{**} = \frac{P^{**}\tau^{**}}{c - b}$  and  $c - b^{**} = \frac{2dR(1 + b^{**})}{q(\tau^{**} + 1)(\tau^{**} + 2)}$  which we derived as we solved for  $b^{**}$ .

$$n^{**} = R \frac{\tau^{**}}{\tau^{**} + 1} \frac{q(\tau^{**} + 1)(\tau^{**} + 2)}{2dR(1 + b^{**})} = \frac{q}{2d} \frac{\tau^{**}(\tau^{**} + 2)}{1 + b^{**}}$$

With similar algebra for the competitive equilibrium we can derive the following  $n = \frac{q}{2d} \frac{\tau}{1 + b}$ .

$$\frac{\tau^{**}(\tau^{**} + 2)}{1 + b^{**}} = \frac{(\eta - 1)(\eta + 1)}{1 + b^{**}} = \frac{\eta^2 - 1}{1 + b^{**}} < \frac{\eta^2 - 1}{1 + b} = \frac{\tau + 1}{1 + b} \text{ since } b^{**} > b$$

Therefore,  $n > n^{**}$ .

**Part 2:**  $n^{**} > n^*$ . For the second part of this lemma, we will use the fact that  $P^{**} > P^*$  as proven by Lemma 3 above. Take the equation 70 that gives the square of the price ratios in these two cases and replace  $P = \frac{R}{\tau + 1}$  to obtain:

$$\left(\frac{P^{**}}{P^*}\right)^2 = \left(\frac{\tau^* + 1}{\tau^{**} + 1}\right)^2 \equiv \kappa = \frac{2dR + q(\tau^* + 1)^2}{2dR + q(\tau^* + 1)} \quad (81)$$

$$n^{**} = \frac{\tau^{**}}{\tau^{**}+1} \frac{2dR+q(\tau^{**}+1)(\tau^{**}+2)}{2d(1+c)} \text{ and } n^* = \frac{\tau^*}{\tau^*+1} \frac{2dR+q(\tau^*+1)}{2d(1+c)}$$

$$n^{**} - n^* = \frac{1}{2d(1+c)} \left\{ \frac{\tau^{**}(\tau^*+1)[2dR+q(\tau^{**}+1)(\tau^{**}+2)] - \tau^*(\tau^{**}+1)[2dR+q(\tau^*+1)]}{2d(1+c)(\tau^*+1)(\tau^{**}+1)} \right\} \quad (82)$$

The numerator can be simplified as follows

$$2dR \underbrace{[\tau^{**}(\tau^*+1) - \tau^*(\tau^{**}+1)]}_{\tau^{**}-\tau^*} + q(\tau^*+1)(\tau^{**}+1) \underbrace{[\tau 2(\tau^{**}+2) - \tau^*]}_{(\tau^{**}+1)^2 - (\tau^*+1)} \quad (83)$$

From 81 we have  $(\tau^{**}+1)^2 = (\tau^{**}+1)^2 \frac{2dR+q(\tau^*+1)}{2dR+q(\tau^*+1)^2}$ , plugging this into the last part of numerator  $(\tau^{**}+1)^2 - (\tau^*+1) = (\tau^*+1) \left\{ (\tau^*+1) \frac{2dR+q(\tau^*+1)}{2dR+q(\tau^*+1)^2} - 1 \right\} = \frac{(\tau^*+1)2dR\tau^*}{2dR+q(\tau^*+1)^2}$

Plugging back to 83 we get  $2dR[\tau^{**} - \tau^*] + \frac{q(\tau^*)(\tau^{**}+1)(\tau^*+1)\tau^*2dR}{2dR+q(\tau^*+1)^2}$

We want this equation to be positive, which requires

$$\begin{aligned} \frac{2dRq\tau^*(\tau^*+1)^2(\tau^{**}+1)}{2dR+q(\tau^*+1)^2} &> 2dR(\tau^* - \tau^{**}) \\ \implies q\tau^*(\tau^*+1)^2(\tau^{**}+1) &> 2dR(\tau^* - \tau^{**}) + q(\tau^* - \tau^{**})(\tau^*+1)^2 \\ \implies (\tau^*+1)^2q \underbrace{[\tau^*(\tau^{**}+1) - \tau^* + \tau^{**}]}_{=\tau^{**}(\tau^*+1)} &> 2dR(\tau^* - \tau^{**}) \end{aligned}$$

We want to show that  $(\tau^*+1)^2q\tau^{**}(\tau^*+1) > 2dR(\tau^* - \tau^{**})$  holds.

$$(\tau^{**}+1)^2 = \eta^2 = \frac{2dR(\tau^*+1)^2+q(\tau^*+1)^3}{2dR+q(\tau^*+1)^2} \text{ and } \tau^{**} = \eta - 1 \implies \tau^{**}(\tau^{**}+2) = \eta^2 - 1 \implies \eta^2 = \tau^{**}(\tau^{**}+2) + 1$$

$$\begin{aligned} \eta^2 = \tau^{**}(\tau^{**}+2) + 1 &= \frac{2dR(\tau^*+1)^2 + q(\tau^*+1)^3}{2dR+q(\tau^*+1)^2} \\ 2dR\eta^2 + q(\tau^*+1)^2\eta^2 &= 2dR(\tau^*+1)^3 \\ 2dR[\eta^2 - (\tau^*+1)^2] &= q(\tau^*+1)^3 - q(\tau^*+1)^2\eta^2 = q(\tau^*+1)^2[\tau^*+1 - \eta^2] \\ 2dR &= \frac{q(\tau^*+1)^2[\tau^*+1 - \eta^2]}{\eta^2 - (\tau^*+1)^2} \end{aligned}$$

$$\begin{aligned} (\tau^*+1)^3q\tau^{**} - 2dR(\tau^* - \tau^{**}) &= (\tau^*+1)^3q\tau^{**} - 2dR[(\tau^*+1) - (\tau^{**}+1)] \\ \implies \frac{\tau^*+1}{\tau^{**}+1}(\tau^*+1)^2q\tau^{**} + 2dR - 2dR\frac{\tau^*+1}{\tau^{**}+1} & \\ = \frac{\tau^*+1}{\tau^{**}+1}[(\tau^*+1)^2q\tau^{**} - 2dR] + 2dR & \end{aligned}$$

Let's focus on  $q\tau^{**}(\tau^* + 1)^2 - 2dR$  because the remaining term is positive

$$\begin{aligned} q\tau^{**}(\tau^* + 1)^2 &> 2dR = \frac{q(\tau^* + 1)^2[\tau^* + 1 - \eta^2]}{\eta^2 - (\tau^* + 1)^2} \\ q(\tau^* + 1)^2\left[\tau^{**} - \frac{\tau^* + 1 - \eta^2}{\eta^2 - (\tau^* + 1)^2}\right] &= q(\tau^* + 1)^2\left[\tau^{**} + \frac{\eta^2 - (\tau^* + 1)}{\eta^2 - (\tau^* + 1)^2}\right] \\ \implies q\tau^{**}(\tau^* + 1)^2 - 2dR &= q(\tau^* + 1)^2\left[\eta - 1 + \frac{\eta^2 - (\tau^* + 1)}{\eta^2 - (\tau^* + 1)^2}\right] \end{aligned}$$

$$\frac{\eta^2 - (\tau^* + 1)}{\eta^2 - (\tau^* + 1)^2} > 1 \implies \eta - 1 + \frac{\eta^2 - (\tau^* + 1)}{\eta^2 - (\tau^* + 1)^2} > 0.$$

□

**Lemma 6.**  $1 - \gamma > 1 - \gamma^* > 1 - \gamma^{**}$

*Proof.*

$$1 - \gamma = \frac{c - b}{P} \text{ together with } b^{**} > b^* \text{ and } P^{**} > P^* \implies 1 - \gamma^* > 1 - \gamma^{**}$$

For  $(1 - \gamma) > (1 - \gamma^*)$ , or  $\frac{c-b}{P} > 1$

Given  $b = \frac{qc(\tau+1)-2dR}{2dR+q(\tau+1)} \implies c - b = \frac{2dR(1+c)}{2dR+q(\tau+1)}$ , and similarly  $c - b^* = \frac{2dR(1+c)}{2dR+q(\tau^*+1)}$

$$\frac{\frac{c-b}{P}}{\frac{c-b^*}{P^*}} = \frac{c-b}{c-b^*} \frac{P^*}{P} = \frac{2dP^* + q}{2dP + q} \frac{P^*}{P} > 1$$

The last inequality is due to  $P^* > P$ .

□

**Lemma 7.**  $(1 - \gamma)n > (1 - \gamma^*)n^* > (1 - \gamma^{**})n^{**}$

*Proof.* This is about the total amount of fire sales. Given that the demand function for risky assets in the interim period is downward sloping (continuous and differentiable as well), the prices will be informative about the amount of fire sales.

$$(1 - \gamma)n = \tau = \frac{R}{P} - 1 \text{ and } P^{**} > P^* > P \implies (1 - \gamma^{**})n^{**} < (1 - \gamma^*)n^* < (1 - \gamma)n$$

□

**Proposition 7.** *Bank balance sheet sizes across different regimes are as follows:*

$$n(1 + b) = n^{**}(1 + b^{**}) > n^*(1 + b^*)$$

*Proof.*

$$n_2 = n^{**} = \frac{\tau_2}{\tau_2 + 1} \frac{2dR + q(\tau_2 + 1)(\tau_2 + 2)}{2d(1 + c)} \quad \text{and} \quad n = n_{competitive} = \frac{\tau}{\tau + 1} \frac{2dR + q(\tau + 1)}{2d(1 + c)}$$

$$b_2 = b^{**} = \frac{cq(\tau_2 + 1)(\tau_2 + 2) - 2dR}{2dR + q(\tau_2 + 1)(\tau_2 + 2)} \quad \text{and} \quad b = b_{\text{competitive}} = \frac{cq(\tau + 1) - 2dR}{2dR + q(\tau + 1)}$$

$$\tau = \eta^2 - 1 \quad \text{and} \quad \tau_2 = \eta - 1$$

$$\begin{aligned} \text{size:} \quad n(1 + b) &= \frac{\tau}{\tau + 1} \frac{2dR + q(\tau + 1)}{2d(1 + c)} \frac{q(\tau + 1)(1 + c)}{2dR + q(\tau + 1)} \\ &= \frac{\tau}{\tau + 1} \frac{q(\tau + 1)}{2d} \\ &= \frac{q\tau}{2d} \end{aligned}$$

$$\begin{aligned} \text{size:} \quad n_2(1 + b_2) &= \frac{\tau_2}{\tau_2 + 1} \frac{2dR + q(\tau_2 + 1)(\tau_2 + 2)}{2d(1 + c)} \frac{q(\tau_2 + 1)(\tau_2 + 2)(1 + c)}{2dR + q(\tau_2 + 1)(\tau_2 + 2)} \\ &= \frac{q\tau_2(\tau_2 + 1)}{2d} \end{aligned}$$

So if  $\tau = \tau_2(\tau_2 + 2)$  we are done!  $\tau = \eta^2 - 1$  and  $\tau_2 = \eta - 1 \implies \tau_2(\tau_2 + 2) = (\eta - 1)(\eta + 1) = \eta^2 - 1 = \tau$ . Thus,  $n(1 + b) = n_2(1 + b_2)$ .

Lastly,  $b_2 > b > b_1$  and  $n > n_2 > n_1$  together imply  $n(1 + b) > n_1(1 + b_1)$ , i.e. the bank balance sheet size is smallest under partial regulation!  $\square$