Idea Flows, Economic Growth, and Trade

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Abstract

We provide a theoretical description of a process that is capable of generating growth and income convergence among economies, growth that is accelerated by freer trade. In this model freer trade replaces inefficient domestic producers with more efficient foreign producers. We add to this static effect a theory of endogenous growth where the engine of growth is the flow of ideas. Ideas are assumed to diffuse by random meeting where people get new ideas by learning from the people they do business or compete with. Trade then has a selection effect of putting domestic producers in contact with the most efficient foreign and domestic producers. We analyze the way that trade in goods, and impediments to it, affect this diffusion, above and beyond the standard effects of trade costs. We find that exclusion of a country from trade reduces the productivity growth, with large long term effects. Smaller trade cost have moderate effects on productivity. The theory also provides a new foundation for the Frechet distribution of productivity that is frequently used in quantitative trade models.

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The progress of a society is all the more rapid in proportion
as it is more completely subjected to external influences.

— Henri Pirenne

1 Introduction

The free-trade institutions established after World War II initiated an era of income convergence in the economies of Europe, North America, and East Asia. Comparative studies by Ben-David (1993), Sachs and Warner (1995), Lucas (2009), and others have documented the empirical connections between openness to trade and growth rates. The economics underlying these connections, if there are any, are not well understood. Our basic theories of international trade do not imply that trade liberalization should induce sustained increases in economic growth rates. The gains from trade that they capture are level effects, not growth effects. It is widely and reasonably believed that trade serves as a *vehicle* for technology diffusion[1] but we lack an explicit description of how reductions in trade costs, say, might induce higher growth rates.

In this paper we provide a theoretical description of a process that is capable of generating growth and income convergence among economies, growth that is accelerated by freer trade. Our starting point is a static trade model adapted from Eaton and Kortum (2002) and Alvarez and Lucas (2007). In this model (as in many others) freer trade replaces inefficient domestic producers with more efficient foreign producers. We add to this familiar, static effect a theory of endogenous growth in which people get new, production-related ideas by learning from the people they do business or compete with. Trade then has a selection effect of putting domestic producers in contact with the most efficient (subject to trade costs) foreign and domestic producers. The identification and analysis of these selection and learning effects is the new contribution of the paper.

Though constructed from familiar components, our model has a complicated, somewhat novel structure, and it will be helpful to introduce enough notation to describe this structure before outlining the rest of the paper. There are n countries, $i = 1, ..., n$, with given populations L_i and given iceberg trade costs, κ_{ij}. There are many goods produced in each country.

[1] It is certainly not the *only* vehicle: Think of the diffusion of nuclear weapons capabilities.

Productivity of any good produced in i will be modeled as a draw from a country-specific probability distribution, defined by its right cdf

$$F_i(z) = \Pr\{\text{cost in } i \text{ of good drawn at random } \geq z\}.$$

We treat populations and trade costs as parameters and analyze the dynamics of the technology profiles $F = (F_1, ..., F_n)$ that serve as the state variables of the model. There are two steps in this analysis.

Given a profile F together with populations and trade costs we define a static competitive equilibrium for the world economy. We use the static model of international trade to determine the way a given technology profile F defines a pattern of world trade, including listings of which sellers in any country are domestic producers or exporters from abroad.

The second step in the analysis is based on a model of technology diffusion that involves stochastic meetings of individual people—we call them *product managers*— who exchange production-related ideas. We use a variation on the Kortum (1997) model, as developed in Alvarez et al. (2008). In this diffusion model, product managers in country i meet managers from some *source distribution* G_i at a given rate α_i and improve their own knowledge whenever such meetings put them in contact with someone who knows more than they do. In our application, this source distribution G_i is the technology profile of the set of sellers who are active in country i, as determined by the trade theory applied in step 1. Under autarchy, then, the source distribution is simply the distribution F_i of domestic producers.[2] Trade improves on this source distribution by replacing some inefficient domestic sellers with more efficient foreigners, replacing F_i with a distribution G_i that stochastically dominates it. It is this selection effect that provides the link between trade volumes and productivity growth that we are seeking.

Technically, trade theory provides a map from a technology profile F to a profile $G = (G_1, ..., G_n)$ of source distributions. The diffusion model gives us a map from each pair (F_i, G_i) into a rate of change $\partial F_i(x,t)/\partial t$. Combining these two steps yields a law of motion for the technology profile F of all n countries together.

The organization of the rest of the paper is as follows. Section 2 introduces our model of technological change in the context of a closed economy. For this case we present a complete characterization of the dynamics of a

[2]Kortum (1997) calls this distribution the technology frontier.

single economy that introduces many features that will be important in understanding the more general case. Section 3 develops the static trade theory that maps a technology profile F into a pattern of world trade. Section 4 then integrates the dynamics of technological change and static equilibrium implied by trade theory. We characterize the balanced growth path for a world economy under constant trade costs and populations, and obtain additional results for the case of costless trade. As we could predict on the basis of the static trade theory alone, a full analytical characterization of the dynamics in the general case is not a possibility, so we continue with numerical results.

In Section 5 we carry out some quantitative explorations to illustrate the effects of trade costs on income levels and growth rates. We calculate equilibrium paths for a symmetric world economy under different trade paths and compare the static and dynamic effects of tariff reductions in this context. We then consider catch-up growth when a small, poor, open economy is introduced into the otherwise symmetric world. These results are illustrated graphically. Section 6 provides a brief discussion of some substantive conclusions suggested by these exercises and of directions for future work that they suggest.

2 Technology Diffusion in a Closed Economy

We begin with a description of technology diffusion and growth in a closed economy. Consumers have identical preferences over a $[0, 1]$ continuum of goods. We use $c(s)$ to denote the consumption of an agent of each of the $s \in [0, 1]$ goods for each period t. There is no intertemporal technology to transfer goods between periods. The period t utility function is given by

$$C = \left[\int_0^1 c(s)^{1-1/\eta} ds \right]^{\eta/(\eta-1)},$$

so goods enter in a symmetrical and exchangeable way. Each consumer is endowed with one unit of labor, which it supplies inelastically.

Each good s is produced with a labor-only, linear technology

$$y(s) = \frac{l(s)}{z(s)} \tag{1}$$

where $l(s)$ is the labor input and $z(s)$ is the cost (labor requirement) associated with good s.

Using the symmetry of the utility function, we group goods by their costs z and write the time t utility as

$$C(t) = \left[\int_{\mathbb{R}_+} c(z)^{1-1/\eta}\ f(z,t)\ dz\right]^{\eta/(\eta-1)}, \qquad (2)$$

where $c(z)$ is the consumption of any good s that has cost z and $f(\cdot,t)$ is the density of costs. We assume that f is continuous. We use $F(z,t)$ for the right cdf of cost, the fraction of goods with cost *higher* than z at time t, so that the cost density is $f(z,t) = -\partial F(z,t)/\partial z$.

In a competitive equilibrium the price of any good z will be $p(z) = wz$ and the ideal price index for the economy at date t is

$$p(t) = \left[\int_{\mathbb{R}_+} p(z)^{1-1/\eta} f(z,t) dz\right]^{\eta/(\eta-1)}. \qquad (3)$$

Real per capita GDP $y(t)$ equals the real wage $w/p(t)$ or

$$y(t) = \left[\int_{\mathbb{R}_+} z^{1-1/\eta} f(z,t) dz\right]^{-\eta/(\eta-1)}. \qquad (4)$$

To ensure convergence of this integral, we require that the left tail of the distribution of cost z goes to zero at a fast enough rate relative to the elasticity of substitution η. See Appendix A.1 for a discussion of this condition.

The study of the dynamics of the closed economy is thus reduced to the study of the evolution of the cost distribution $F(z,t)$: technological diffusion.

We model technological diffusion as a process of search and matching involving product managers of the $s \in [0,1]$ goods, with one manager per good. We can think of these managers as a negligible subset of agents or as a function that some agents perform in addition to supplying labor. Either way, it is an activity that requires no time and earns no private return. We assume these managers interact with each other and exchange production-related ideas. Meetings occur at a rate α per unit of time. Each meeting is a random draw from the population of managers of all goods. When a manager with cost z meets another with cost $z' < z$ he adopts z' for the production of his own good. We assume that the diffusion of technology is the same across any two goods, no matter how different they are.[3] While we refer to

[3]Perhaps a more descriptive, yet less tractable model will distinguish between goods that are similar, in terms of how transferable is the technology.

this process as technology *diffusion*, it might as well be called *innovation*, since the more advanced technology used for one good has to be adapted to a different good. The effect of indirect links, of the role of chance in our diffusion process, is familiar to us from the history of technology.[4] Next we give a mathematical description of this process.

To motivate a law of motion for the cost distribution $F(z,t)$, we describe the discrete change between t and $t+\Delta$, and then derive its continuous-time limit. For a given level of the cost z at date t, we assume that

$$\begin{aligned} F(z,t+\Delta) &= \Pr\{\text{cost } > z \text{ at } t+\Delta\} \\ &= \Pr\{\text{cost } > z \text{ at } t\} \times \Pr\{\text{no lower draw in } (t,t+\Delta)\} \\ &= F(z,t)F(z,t)^{\alpha\Delta}. \end{aligned}$$

The first term in the right hand side reflects the option, which managers always have, to continue with their current cost. The second is the probability that in $\alpha\Delta$ randomly drawn meetings an agent with cost z does not meet anyone with a lower cost. Given our assumption of independent draws, the fraction of managers with cost above z at date $t+h$ is given by product of these two terms. Now we take the limit as $\Delta \to 0$ to obtain:[5]

$$\frac{1}{F(z,t)}\frac{\partial F(z,t)}{\partial t} = \alpha \log\left(F(z,t)\right). \tag{5}$$

Then for any initial distribution (right cdf) $F(z,0)$ the path of F is given by

$$\log(F(z,t)) = \log(F(z,0))e^{\alpha t}. \tag{6}$$

It is evident from (6) that the law of motion (5) implies a non-decreasing, level of real income $y(t)$. For empirical reasons, our interest is in sustained growth of economies that either grow at a fairly constant rate or will do so asymptotically. A central construct in our analysis will therefore be a

[4]Here is a nice example, taken from chapter 13 of Diamond (1998): "[N]ew technologies and materials make it possible to generate still other new technologies by recombination ... Gutenberg's press was derived from screw presses in use for making wine and olive oil, while his ink was an oil-based improvement on existing inks...."

[5]A similar, but not identical, differential equation could be based on the more familar assumption of Poisson arrivals, as opposed to the continuous arrivals postulated here. The formulation here has the convenient property of preserving distyributions in the Weibull family. See Alvarez *et al.* (2008).

balanced growth path (BGP), defined as a right cdf $\Phi(z)$ (with continuous density $\phi = -\ \Phi'(z)$) and a growth rate $\nu > 0$ such that

$$F(z,t) = \Phi(e^{vt}z) \quad \text{for all} \quad t \geq 0$$

is a solution to (5). Then on a BGP

$$f(z,t) = -\frac{\partial F(z,t)}{\partial z} = \phi(e^{vt}z)e^{vt}$$

holds. Real GDP is

$$\begin{aligned} y(t) &= \left[\int_{\mathbb{R}_+} z^{1-1/\eta}\phi(e^{vt}z)e^{vt}dz\right]^{-\eta/(\eta-1)} \\ &= e^{vt}\left[\int_{\mathbb{R}_+} x^{1-1/\eta}\phi(x)dx\right]^{-\eta/(\eta-1)} \end{aligned} \tag{7}$$

provided the integral on the right converges. In the rest of this section we (i) characterize all pairs (Φ, ν) that are balanced growth paths and (ii) characterize the initial distributions $F(z,0)$ from which the solution $F(e^{-vt}z,t)$ will converge asymptotically to $\Phi(z)$.

The possible balanced growth solutions to (5) are contained in the Weibull family of distributions, a two-parameter family defined by the right cdfs:

$$F(z,0) = \exp(-\lambda z^{1/\theta}), \quad \theta, \lambda > 0. \tag{8}$$

A Weibull random variable $\tilde{z}$ is an exponentially distributed random variable raised to the power θ, since

$$\begin{aligned} \Pr\{\tilde{z} &\geq z\} = \Pr\{\tilde{z}^\theta \geq z^\theta\} \\ &= F(z^\theta, 0) \\ &= \exp(-\lambda z). \end{aligned}$$

We have

Proposition 1. The cdf/growth rate pair (Φ, ν) is a balanced growth path of (5) if and only if Φ is a Weibull distribution with parameters $\lambda > 0$ and $\theta = \nu/\alpha$.

Proof: It is immediate from (5) that if $F(z,0)$ has a Weibull distribution with parameters λ and θ then

$$\log\left(F(z,t)\right) = -\lambda e^{\alpha t} z^{1/\theta}$$

which is Weibull with parameters $\lambda(t) = \lambda e^{\alpha t}$ and θ. Then if $\Phi(e^{vt}z) = F(z,t)$,

$$\begin{aligned} \log(\Phi(z)) &= \log\left(F(e^{-\nu t}z, t)\right) \\ &= -\lambda e^{\alpha t}\left(e^{-\nu t}z\right)^{1/\theta} \\ &= -\lambda z^{1/\theta} \end{aligned}$$

if and only if $\nu = \alpha\theta$. Thus $(\Phi, \alpha\theta)$ is a BGP.

Conversely, suppose (Φ, ν) is a BGP, so that $F(z,t) = \Phi(e^{vt}z)$ solves (5). Then

$$\log(\Phi(e^{vt}z)) = \log(\Phi(z))e^{\alpha t}$$

Differentiating both sides with respect to t :

$$\frac{1}{\Phi(e^{vt}z)}\Phi'(e^{vt}z)\nu e^{vt}z = \alpha \log(\Phi(z))e^{\alpha t}$$

Letting $\theta = \nu/\alpha$ and evaluating at $t = 0$ gives:

$$\frac{\Phi'(z)}{\Phi(z)}\theta z = \log(\Phi(z)). \tag{9}$$

For any constant $\lambda > 0$, the unique solution to (9) that satisfies the boundary condition $\log(\Phi(1)) = \lambda$ is

$$\Phi(z) = \exp\left(-\lambda z^{1/\theta}\right).$$

This proves that if (Φ, ν) is a BGP, Φ must be a Weibull distribution with parameters $\theta = \nu/\alpha$ and any $\lambda > 0$.□

We next address the stability of balanced growth paths: Under what conditions on the initial distribution $F(z,0)$ will it be the case that

$$\lim_{t\to\infty} \log\left[F(e^{-\alpha\theta t}z, t)\right] = -\lambda z^{1/\theta} \quad \text{for all } \; z > 0 \tag{10}$$

for some $\lambda > 0$ and $\theta > 0$? We seek an answer to this question for all pairs $(\lambda, \theta) \in \mathbb{R}^2_{++}$ since all such pairs define a Weibull distribution and hence by Proposition 1 a balanced growth path, and for all initial distributions $F(\cdot, 0)$ on $\mathbb{R}_{++}$, since all define a path (6). We begin with

Proposition 2. Suppose that for some $\lambda > 0$ and $\theta > 0$ the initial distribution $F(., 0)$ satisfies

$$\lim_{x \to 0} \theta f(x^\theta, 0) x^{\theta - 1} = \lambda. \tag{11}$$

Then the solution (6) to (5) satisfies (10). Conversely, for any $\lambda > 0$, condition (11) defines the value of θ such that (10) holds for the pair (λ, θ).

Proof.

We can use (6) and the change of variable $x = z^{1/\theta}$ to restate (10) as

$$\lim_{t \to \infty} \log(F(e^{-\alpha\theta t} x^\theta), 0) e^{\alpha t} = -\lambda x \quad \text{for all} \;\; x > 0.$$

Defining $y = e^{-\alpha t} x$, we obtain the equivalente expression

$$\lim_{y \to 0} \frac{\log(F(y^\theta, 0))}{y} = -\lambda.$$

From fact that the left hand side is the definition of the derivative at zero of the function $\log(F(y^\theta, 0))$, and applying l'Hospital's rule, it follows that

$$\lim_{y \to 0} \theta y^{\theta - 1} f(y^\theta, 0)) = \lambda.$$

Then the result follows from (11). Since each step is an equivalence, the converse holds as well. $\square$

Proposition 2 and (6) imply an immediate

Corollary: Equation (11) holds for $\lambda > 0$ and $\theta > 0$ if and only if

$$\lim_{x \to 0} \theta x^{\theta - 1} f(x^\theta, t) = \lambda e^{\alpha t} \tag{12}$$

holds for all $t \geq 0$.

It is sometimes useful to restate (11) in the equivalent form

$$\lim_{y \to 0} \frac{f(y, 0)}{(\lambda/\theta)\, y^{1/\theta - 1}} = 1,$$

where $y = z^\theta$. We can say that in the left tail of the distribution, the density $f(y, 0)$ is equivalent to the power function $(\lambda/\theta)\, y^{1/\theta - 1}$.

The following result relates θ to the elasticity at 0 of the initial cdf $1 - F(z, 0)$.

Proposition 3. Suppose that the density of the right cdf $F(z, 0)$ satisfies property (11) for some $\theta > 0$ and $\lambda > 0$. Then

$$\lim_{z \to 0} \frac{f(z,0)z}{1 - F(z,0)} = \frac{1}{\theta}. \tag{13}$$

Proof. Rearranging equation (11) we obtain

$$\begin{aligned}
\lambda &= \lim_{z\to 0} \theta \frac{1 - F(z^\theta, 0)}{z} \frac{f(z^\theta, 0) z^\theta}{1 - F(z^\theta, 0)} \\
&= \theta \lim_{z\to 0} \frac{1 - F(z^\theta, 0)}{z} \lim_{z\to 0} \frac{f(z^\theta, 0) z^\theta}{1 - F(z^\theta, 0)} \\
&= \theta \lim_{z\to 0} \theta f(z^\theta, 0) z^{\theta - 1} \lim_{z\to 0} \frac{f(z^\theta, 0) z^\theta}{1 - F(z^\theta, 0)} \\
&= \theta\lambda \lim_{z\to 0} \frac{f(z^\theta, 0) z^\theta}{1 - F(z^\theta, 0)}.
\end{aligned}$$

The second but last equality follows from applying l'Hospital's rule to the first limit, and the last equality follows from condition (11). Thus, equation (13) follows from the last line. □

We note that (12) and Proposition 3 imply that

$$\lim_{z\to 0} \frac{f(z,t)z}{1 - F(z,t)} = \frac{1}{\theta}$$

holds for all $t \geq 0$.

In Proposition 1 we showed that every point (λ, θ) in the interior of $\mathbb{R}^2_+$ defines a Weibull distribution that in turn defines a balanced growth path for the paths (6) and that all balanced paths of (6) can be defined by such a point. In Propositions 2 and 3 we showed that all initial distributions $F(., 0)$ that satisfy (11) and (13) for some pair (λ, θ) in the interior of $\mathbb{R}^2_+$ define paths (6) that converge to a Weibull distribution with the parameters (λ, θ) in the sense of (10). That is, we have characterized the basin of attraction of all balanced growth paths.

There are, of course, initial distributions that generate paths that do not converge in the sense of (10): any distribution with a support that is bounded away from 0, for example. A log normal $F(.,0)$ has an elasticity of $1 - F(.,0)$ that converges to ∞ and so equation (13) implies $\theta = 0$. In this case, the economy does not have a balanced growth path with strictly positive growth. In the opposite extreme, an example of a distribution with an elasticity converging to zero is

$$1 - F(z,0) = \exp\left[-\sum_{i=1}^{\infty}(\frac{\beta}{\delta})^i(1 - z^{\delta^i})\right],\ z \in [0,1],\ 0 < \delta < \beta < 1.$$

The elasticity of $1 - F(z,0)$ equals $\sum_{i=1}^{\infty} \beta^i z^{\delta^i}$, and therefore, it tends to 0 as $z \to 0$. In this case the economy does not have a balanced growth path since the growth rate will be increasing without bound as time passes.

Initial distributions that satisfy (4) but fail to satisfy (3) can also be constructed. One example is

$$1 - F(z,0) = z\exp\left[-\sum_{i=1}^{\infty}(\frac{\beta}{\delta})^i(1 - z^{\delta^i})\right],\ z \in [0,1],\ 0 < \delta < \beta < 1.$$

The elasticity of $1 - F(z,0)$ equals $1 + \sum_{i=1}^{\infty} \beta^i z^{\delta^i}$, and therefore tends to 1 as $z \to 0$, but this cdf does satisfy condition (11). In this case, $\lim_{z\to 0}(1-F(z^{\theta},0))/z = 0\ (\infty)$ for all $\theta \leq (>)1$, which implies that condition (11) is not satisfied for any θ.

3 Static Trade Model

We consider a world economy consisting of n countries, indexed by $i = 1, ..., n$. Each country under autarky is idenical to the closed economy described in Section 2. We use the same notation here, adding the country subscript i to the variables $c_i(s)$, $z_i(s)$, $y_i(s)$, and ℓ_i. In this many country case we group goods by the profile $\mathbf{z} = (z_1, ..., z_n)$ of cost across the

n locations, where z_i is the labor required to produced this good in location i, as in Alvarez and Lucas (2007). The function $F_i(z,t)$ is the right c.d.f. of cost in country i, and $f_i(z,t) = -\partial F_i(z,t)/\partial z$ is the associated density. We assume that production costs are independently distributed across countries. We use unsubscripted F and f for the joint world distribution: $F(\mathbf{z},t) = \prod_{i=1}^{n} F_i(z_i,t)$ and $f(\mathbf{z},t) = \prod_{i=1}^{n} f_i(z_i,t)$. With this notation we can write the time t utility as

$$C_i(t) = \left[\int_{\mathbb{R}_+^n} c_i(\mathbf{z})^{1-1/\eta} \; f(\mathbf{z},t) d\mathbf{z} \right]^{\eta/(\eta-1)},$$

where $c_i(\mathbf{z})$ is the consumption in country i of the good s that has cost profile $\mathbf{z}$.

We use $w_i(t)$ for the time t wages in i in units of time t numeraire. We assume iceberg shipping costs: when a good is sent from country k a fraction κ_{ik} of the good arrives in i. The costs κ_{ik} are the same for all goods, and the $\kappa_{ii} = 1$ for all i. Each good $\mathbf{z} = (z_1, ..., z_n)$ is available in i at the unit prices

$$\frac{z_1 w_1(t)}{\kappa_{i1}}, ..., \frac{z_n w_n(t)}{\kappa_{in}},$$

which reflect both production and transportation costs.

We let $p_i(\mathbf{z},t)$ be the prices paid for good $\mathbf{z}$ in i at t. Then

$$p_i(\mathbf{z},t) = \min_j \left[\frac{w_j(t)}{\kappa_{ij}} z_j \right] \tag{14}$$

since agents in i buy the good at the lowest price. Given prices $p_i(\mathbf{z},t)$, the ideal price index at t is the minimum cost of providing one unit of aggregate consumption $C_i(t)$:

$$p_i(t) = \left[\int_{\mathbb{R}_+^n} p_i(\mathbf{z},t)^{1-\eta} \; f(\mathbf{z},t) d\mathbf{z} \right]^{1/(1-\eta)}. \tag{15}$$

We turn to the description of the equilibrium of the trade model for given cdfs $F(\cdot,t)$. We assume that there is no borrowing and lending: trade balance must hold in each period. Because of the static nature of the equilibrium we omit the index t in its description.

Define the sets $\mathbf{B}_{ij} \subset \mathbb{R}_+^n$ by

$$\mathbf{B}_{ij} = \left\{ \mathbf{z} \in \mathbb{R}_+^n : p_i(\mathbf{z}) = \frac{w_j}{\kappa_{ij}} z_j \right\}.$$

That is, $\mathbf{B}_{ij}$ is the set of goods that people in i want to buy from producers in j. For each i, $\cup_j \mathbf{B}_{ij} = \mathbb{R}_+^n$. Note that the sets $\mathbf{B}_{ij}$ are all cones, defined equivalently by

$$\mathbf{B}_{ij} = \left\{ \mathbf{z} \in \mathbb{R}_+^n : \frac{z_j w_j}{\kappa_{ij}} \leq \frac{z_k w_k}{\kappa_{ik}} \quad \text{for all } k \right\}.$$

The price index p_i must be calculated country by country. We have

$$\Pr\left\{ \frac{z_j w_j}{\kappa_{ij}} \leq \frac{z_k w_k}{\kappa_{ik}} \mid z_j \right\} \Pr\{z_j\} = \Pr\left\{ z_k \geq \frac{\kappa_{ik} w_j}{\kappa_{ij} w_k} z_j \mid z_j \right\} \Pr\{z_j\}$$

$$= F_k \left(\frac{\kappa_{ik} w_j}{\kappa_{ij} w_k} z_j \right) f_j(z_j) dz_j$$

Then

$$\begin{aligned} p_i^{1-\eta} &= \int p_i(\mathbf{z})^{1-\eta} f(\mathbf{z}) d\mathbf{z} = \sum_{j=1}^n \int_{\mathbf{B}_{ij}} \left(\frac{z_j w_j}{\kappa_{ij}} \right)^{1-\eta} f(\mathbf{z}) d\mathbf{z} \\ &= \sum_{j=1}^n \int_0^\infty \left(\frac{z_j w_j}{\kappa_{ij}} \right)^{1-\eta} f_j(z_j) \prod_{k \neq j} F_k \left(\frac{\kappa_{ik} w_j}{\kappa_{ij} w_k} z_j \right) dz_j \end{aligned}$$

Letting

$$a_{ijk} = \frac{w_j \kappa_{ik}}{w_k \kappa_{ij}} \ .$$

we have

$$p_i^{1-\eta} = \sum_{j=1}^n \left(\frac{w_j}{\kappa_{ij}} \right)^{1-\eta} \int_0^\infty z_j^{1-\eta} f_j(z_j) \prod_{k \neq j} F_k(a_{ijk}\ z_j)\ dz_j\ . \tag{16}$$

Consumption of good $\mathbf{z}$ in country i equals

$$c_i(\mathbf{z}) = \left(\frac{p_i}{p_i(\mathbf{z})} \right)^\eta C_i = \left(\frac{p_i}{p_i(\mathbf{z})} \right)^\eta \frac{w_i L_i}{p_i}.$$

where the first equality follows from individual maximization and the second follows from the budget constraint $p_i C_i = w_i L_i$ since we have assumed that trade is balanced in each period. The derived demand for labor in country i is thus

$$\begin{aligned}
\sum_{j=1}^{n} \int_{\mathbf{B}_{ji}} c_j(\mathbf{z}) \frac{z_i}{\kappa_{ji}} f(\mathbf{z}) d\mathbf{z} &= \sum_{j=1}^{n} \int_{\mathbf{B}_{ji}} \left(\frac{p_j}{p_j(\mathbf{z})} \right)^{\eta} \frac{w_j L_j}{p_j} \frac{z_j}{\kappa_{ji}} f(\mathbf{z}) d\mathbf{z} \\
&= \sum_{j=1}^{n} \int_{0}^{\infty} \left(\frac{p_j}{p_j(\mathbf{z})} \right)^{\eta} \frac{w_j L_j}{p_j} \frac{z_i}{\kappa_{ji}} f_i(z_i) \prod_{k \neq i} F_k \left(\frac{w_i \kappa_{jk}}{w_k \kappa_{ji}} \, z_i \right) \, dz_i \\
&= \sum_{j=1}^{n} \int_{0}^{\infty} \left(\frac{\kappa_{ji} p_j}{w_i z_i} \right)^{\eta} \frac{w_j L_j}{p_j} \frac{z_i}{\kappa_{ji}} f_i(z_i) \prod_{k \neq i} F_k \left(\frac{w_i \kappa_{jk}}{w_k \kappa_{ji}} \, z_i \right) \, dz_i .
\end{aligned}$$

Since labor is supplied inelastically, this implies

$$L_i = \sum_{j=1}^{n} \left(\frac{p_j}{w_i} \right)^{\eta} \frac{w_j L_j}{p_j} \kappa_{ji}^{\eta-1} \int_{0}^{\infty} z_i^{1-\eta} f_i(z_i) \prod_{k \neq i} F_k \left(\frac{w_i \kappa_{jk}}{w_k \kappa_{ji}} \, z_i \right) \, dz_i \; . \tag{17}$$

For given trade costs, (16) expresses prices as a function $p(w)$ of wages so we can substitute into (17) to obtain the excess demand functions

$$Z_i(w) = \sum_{j=1}^{n} \left(\frac{p_j(w)}{w_i} \right)^{\eta} \frac{w_j L_j}{p_j(w)} \kappa_{ji}^{\eta-1} \int_{0}^{\infty} z_i^{1-\eta} f_i(z_i) \prod_{k \neq i} F_k \left(\frac{w_i \kappa_{jk}}{w_k \kappa_{ji}} \, z_i \right) \, dz_i - L_i. \tag{18}$$

as n equations in $w = (w_1, ..., w_n)$, given populations L, trade costs K, and the distributions $F = (F_1, ..., F_n)$. Unless otherwise stated, we assume that $L_i > 0$ and that $0 < \kappa_{ij} \leq 1$.

Definition. A *static equilibrium* is a wage vector $w = (w_1, ..., w_n) \in \mathbb{R}^n_+$ such that $Z_i(w) = 0$.

The next proposition shows that a static equilibrium exists and that, provided that $\eta \geq 1$, the excess demand Z has the gross substitute property. This establishes that there is a unique static equilibrium, which is easily solvable by a tatonnement process, and satisfies the natural comparative statics with respect to population sizes.

Proposition 4: We take as given trade costs K, populations L, and distributions F. We assume that $L_i > 0$ and that $0 < \kappa_{ij} \leq 1$ and that the right cdfs $F = (F_1, ..., F_n)$ have continuous densities and satisfy

$$\lim_{z \to 0} \frac{f_i(z) z}{1 - F_i(z)} = \frac{1}{\theta_i} > \eta - 1$$

for all $i = 1, ..., n$. Then there exists a static equilibrium wage w. Moreover, if $\eta > 1$, the excess demand system has the gross substitute property, and hence (i) the static equilibrium wage w is unique, and (ii) equilibrium relative wages are decreasing in population sizes:

$$\frac{\partial (w_j / w_i)}{\partial L_i} > 0$$

for all $j \neq i$.

Proof: To establish existence we show that the the excess demand system satisfies i) Walras' law, i.e. $\sum_{i=1}^n w_i Z_i(w) = 0$ for all w, ii) that the functions Z are continuous and homogenous of degree zero in w, iii) that $Z(w)$ are bounded from below, and iv) that $\max_j Z(w) \to \infty$ as $w \to w^0$ where w^0 is on the boundary of the n dimensional simplex.

Part (i) follows from replacing p_i in the expression for Z_i, (ii) continuity is immediate since the functions F_i are differentiable, and homogeneity is immediate by inspection of (16) and (17). For (iii), we can take $-\max_j L_j$ to be the lower bound. For (iv) we assume, without loss of generality, that $0 = w_1^0 \leq w_2^0 \leq \cdots \leq w_n^0 = 1$, and show that $Z_1(w) \to +\infty$. For any w we have

$$\begin{aligned} & Z_1(w) - L_1 \\ \geq \; & \left(\frac{w_n}{w_1}\right)^{\eta} \left(\frac{w_n}{p_n(w)}\right)^{1-\eta} L_n \, \kappa_{n1}^{\eta-1} \int_0^{\infty} z_1^{1-\eta} f_1(z_1) \prod_{k \neq 1} F_k \left(\frac{w_1 \kappa_{nk}}{w_k \kappa_{n1}} \; z_1 \right) dz_1 \end{aligned}$$

Note that for all i we have

$$p_i \leq (w_n / \kappa_{in}) \left[\int_0^{\infty} z^{1-\eta} f_n(z) \; dz \right]^{1/(1-\eta)} ,$$

where the left hand side is the price that would be obtained by consumers in country i if they restrict themselves to buy only from country n. Considering $w = w^r$ we have that $w_n / p_n(w)$ is uniformly bounded from above by the previous expression, setting $i = n$ for all r large enough since $w_n^0 = 1$. Finally, for any $\epsilon > 0$, $w_1/w_k \leq 1-\epsilon$ for all r large enough, and hence $F_k\left(\frac{w_1\kappa_{nk}}{w_k\kappa_{n1}}\,z_1\right) > 0$ for all finite z_1. Using that $\eta > 1$ and taking limits we obtain the desired result. Given (i)-(iv), existence of an static trade equilibrium wage follows from Proposition 17.C.1 in Mas-Colell et al. (1995).

To establish the gross substitute property, since the excess demand system satisfies Walras' law, it suffices to show that $\partial Z_i(w)/\partial w_r > 0$ for all $i, r = 1, ..., n$ and $i \neq r$. First notice that $p_j(w)$ is increasing in each of the components of w and homogenous of degree one in w for all j. This implies that $w_r/p_r(w)$ is increasing in w_r. We have:

$$\begin{aligned}
\frac{\partial Z_i(w)}{\partial w_r} &= \sum_{j=1,j\neq r}^{n} \frac{\partial}{\partial w_r}\left[\left(\frac{w_i}{p_j(w)}\right)^{-\eta}\frac{w_j}{p_j(w)}L_j\right]\int_0^\infty \left(\frac{z_i}{\kappa_{ji}}\right)^{1-\eta} f_i(z_i)\prod_{k\neq i}F_k\left(\frac{w_i}{w_k}\frac{\kappa_{ik}}{\kappa_{ij}}z_i\right)\,dz_i \\
&+ \sum_{j=1,j\neq r}^{n}\left(\frac{w_i}{p_j(w)}\right)^{-\eta}\frac{w_j}{p_j(w)}L_j\int_0^\infty \left(\frac{z_i}{\kappa_{ji}}\right)^{1-\eta} f_i(z_i)\prod_{k\neq i}\frac{\partial}{\partial w_r}\left[F_k\left(\frac{w_i}{w_k}\frac{\kappa_{ik}}{\kappa_{ij}}z_i\right)\right]\,dz_i \\
&+ \frac{\partial}{\partial w_r}\left[\left(\frac{w_i}{p_r(w)}\right)^{-\eta}\frac{w_r}{p_r(w)}L_r\right]\int_0^\infty \left(\frac{z_i}{\kappa_{ri}}\right)^{1-\eta} f_i(z_i)\prod_{k\neq i}F_k\left(\frac{w_i}{w_k}\frac{\kappa_{ik}}{\kappa_{ir}}z_i\right)\,dz_i \\
&+ \left(\frac{w_i}{p_r(w)}\right)^{-\eta}\frac{w_r}{p_r(w)}L_r\int_0^\infty \left(\frac{z_i}{\kappa_{ri}}\right)^{1-\eta} f_i(z_i)\prod_{k\neq i}\frac{\partial}{\partial w_r}\left[F_k\left(\frac{w_i}{w_k}\frac{\kappa_{ik}}{\kappa_{ir}}z_i\right)\right]\,dz_i
\end{aligned}$$

For $j \neq r$, using that $\eta > 1$ and $p_j(w)$ in increasing, we get $\frac{\partial}{\partial w_r}\left[\left(\frac{w_i}{p_j(w)}\right)^{-\eta}\frac{w_j}{p_j(w)}L_j\right] > 0$. For $j = r$, using that $\eta > 0$, that $w_r/p(w)$ is decreasing in w_r we get that $\frac{\partial}{\partial w_r}\left[\left(\frac{w_i}{p_r(w)}\right)^{-\eta}\frac{w_r}{p_r(w)}L_r\right] > 0$. For $k = r \neq i$ we have that $\frac{\partial}{\partial w_r}\left[F_k\left(\frac{w_i}{w_k}z_i\right)\right] > 0$ since F_k is decreasing.

That w_i/w_j, relative wages of country i respect to any country j, are decreasing in L_i, follows from the strong gross substitute property. In particular, form an an application of the Hick's law of demand, since the excess demand of country i decreases with L_i, while the excess demand for any other country increases with L_i, –see, for example, first corollary of Theorem 3 in Quirk (1968). □

For future reference we analyze the correspondence between equilibrium wages and population sizes. Under the conditions stated in Proposition 5 there is a unique equilibrium wage w for any population size vector L. The next propositions established the converse: for any given $w \in \mathbb{R}^n_{++} > 0$ there is a unique vector of populations L for which w is a static trade equilibrium.

Proposition 5: For any trade costs K, distributions F, and wages $w \in \mathbb{R}^n_{++}$, there exists a vector populations L so that w is a static trade equilibrium. The population sizes L are unique, up to scale.

Proof: Let w be an equilibrium for L. The excess demand system can we written as

$$w_i \ L_i = \sum_{j=1}^{n} \ w_j \ L_j \ \left(\frac{w_i}{p_j(w)}\right)^{1-\eta} \int_0^\infty \ \left(\frac{z_i}{\kappa_{ji}}\right)^{1-\eta} f_i(z_i) \prod_{k \neq i} F_k \left(\frac{w_i}{w_k}\frac{\kappa_{ik}}{\kappa_{ij}} z_i\right) \ dz_i$$

Let $d_{ij}(w)$ be the strictly positive constant

$$d_{ij}(w) \equiv \left(\frac{w_i}{p_j(w)}\right)^{1-\eta} \int_0^\infty \ \left(\frac{z_i}{\kappa_{ji}}\right)^{1-\eta} f_i(z_i) \prod_{k \neq i} F_k \left(\frac{w_i}{w_k}\frac{\kappa_{ik}}{\kappa_{ij}} z_i\right) \ dz_i$$

which are independent of L. Thus letting $l_i \equiv w_i L_i$ we can write the excess demand system as:

$$l_i = \sum_{j=1}^{n} l_j \ d_{ij}(w)$$

Notice that the definition of $p_i(w)$ implies that: $\sum_{i=1}^{n} d_{ij} = 1$ for all $j = 1, ..,$. By an application of the Perron-Frobenious theorem this linear set of equations has a unique strictly positive solution in the simplex. Let $l(w)$ be the unique solution. Setting $L_i = l_i(w)/w_i$ we obtain the unique values of L that rationalize w as a static trade equilibrium. $\square$

The static multi-country economy that we analyze has constant returns to scale in the production of all goods, which has several implications that we will use in our analysis of technology diffusion. We present two lemmas, which we state without proof. The first implication is that scaling up by a constant the distribution of productivities in *all* countries in the world, there is no change in any relative price.

Corollary 1: Let (w, p) be the equilibrium wages and prices for an economy with K, L, F. Let $\xi \in \mathbb{R}++$ and define F_i^ξ as $F_i^\xi(z) = F_i(\xi z)$, for all i. Then

(w, p) are also the equilibrium wages and prices for an economy with K, L, F^{ξ}.

The second implication of the constant returns to scale is that if we scale *both* the distribution of the costs and population size of a *particular* country i by $\xi_i > 0$, so that if $\xi_i < 1$ the country is more productive, but it has a smaller size. In this case the equilibrium wages are scaled up by $1/\xi_i$ and the prices of aggregate consumption are the same.

Corollary 2: Let (w, p) be the equilibrium wages and prices for an economy with L, K and F. Let $\xi \in \mathbb{R}^n_{++}$, and consider an economy with populations $L^{\xi}_i = L_i \, \xi_i$ and distributions $F^{\xi}_i(\xi_i z) = F_i(z)$ for all i. The economy with (L^{ξ}, K, F^{ξ}) has equilibrium wages and prices given by $w^{\xi}_i = w_i \, / \, \xi_i$ and $p^{\xi}_i = p_i$ for all i .

The next example illustrates the previous results:

Example: Assume $F_i(z) = \exp\left(-\lambda_i \, z^{1/\theta}\right)$ for all i and $\kappa_{ij} = 1$. Direct computations -see Alvarez and Lucas (2007) - gives that equilibrium wages are $w_i = \lambda_i / L_i$. Clearly, relative wages are decreasing in country sizes. Furthermore, each vector L corresponds to an equilibrium vector w. Finally, multiplying the cost distribution by ξ_i is equivalent to consider the value to $\lambda^{\xi}_i = \lambda_i / \xi_i^{1/\theta}$ for the definition of F_i, and since θ is common for all i, in the case where $\xi_i = \xi_j = \xi$ for all i, j all relative wages are independent of ξ.

4 Diffusion in a World Economy

We now turn to the analysis of the dynamics of a world economy, for given fixed values of the trade costs K and equilibrium wage $w(t)$, determined as in Section 3. The state variables are the profile $F = (F_1, ..., F_n)$ of the right cdfs of each of the n countries. The law of motion is

$$\frac{\partial \log(F_i(z,t))}{\partial t} = \alpha_i \log(G_i(z,t)) \tag{19}$$

where the functions $G_i(z, t)$ are the right cdfs of the sellers that are active in i at date t:

$$G_i(z,t) = \sum_{j=1}^{n} \int_z^{\infty} f_j(y,t) \prod_{k \neq j} F_k \left(\frac{w_j \kappa_{ik}}{w_k \kappa_{ij}} y, t \right) dy. \tag{20}$$

The densities corresponding to F_i and G_i are

$$f_i(z,t) = -\frac{\partial F_i(z,t)}{\partial z} \quad \text{and} \quad g_i(z,t) = -\frac{\partial G_i(z,t)}{\partial z}.$$

In Section 3 we defined a static equilibrium. We are now in a position to define an equilibrium that describes the full dynamics a world economy given parameters K, L and initial distributions $F(\cdot, 0) = (F_1(\cdot, 0), ..., F_n(\cdot, 0))$.

Definition. An *equilibrium* is a time path of wages $w(t) = (w_1(t), ..., w_n(t))$ and right cdfs $F(\cdot, t)$ for all $t \geq 0$ such that

(i) $w(t)$ is a static equilibrium as defined in Section 3, and

(ii) given $w(t)$ the path $F(\cdot, t)$ satisfies (19) and (20).

In this section, we simply take a path of wages as given and study the dynamics for the distributions F and G implied by (19) and (20). We note that *given* a wage path the evolution of these distributions will be independent of the elasticity of substitution η and of the country sizes $L = (L_1, ..., L_n)$.

We begin with a proposition that derives properties of the cdfs $G = (G_1, ..., G_n)$ from assumptions on the profile F. In particular, Proposition 6 derives some properties that the left tail (z near 0) of G must have from assumptions on F that mirror assumptions we applied in the study of the closed economy in Section 2, combined with very weak assumptions on trade. Proposition 7 derives the common growth rate on a balanced growth path for the n-country model and other properties that obtain near $z = 0$. Proposition 8 describes the full dynamics for the left tail.

The following condition on the tail behavior these distributions will be used in deriving the various results in this section. It is a natural generalization to case of n locations of the condition required in Section 2 for a distribution to be in the basin of attraction of a balanced growth path, supplemented with a condition on the nature of trade. Where it will not cause confusion, we suppress the time arguments of F and G and their dependence on $\mathbf{w}$ and $\mathbf{K}$.

Condition C: We say that a profile $F(z) = (F_1(z), ..., F_n(z))$ satisfies Condition C if

(C1)

$$\lim_{z \to 0} \frac{z f_i(z)}{1 - F_i(z)} \equiv \frac{1}{\theta_i} > 0 \quad \text{for all } i,$$

(C2) $\lim_{z\to 0}\theta_i x^{\theta_i-1}f_i(x^{\theta_i}) = \lambda_i \geq 0$ for all i, with $\lambda_i > 0$ if $\theta_i = \max_j \theta_j \equiv \theta$, and

(C3) $\kappa_{ij} > 0$ for all i, j.

We note that $\theta x^{\theta-1}f_j(x^\theta)$ is the density function of the random variable $\tilde{x} = \tilde{z}^{1/\theta}$ where $\tilde{z}$ has the density f_j. Thus Condition C requires that the initial cost distributions of the countries with largest tail parameter θ have a strictly positive density at 0. If Condition C holds no country is capable of ever-accelerating growth (C1), at least one country is capable of sustained growth at a positive rate (C2), and all countries are connected in the sense that it is possible for any country to trade with any other country (C3).

The next result shows that the distributions G_i share a common left tail, with an elasticity at zero equal to $1/\theta$, where $\theta \equiv \max_i \theta_i$.

Proposition 6: Assume that the profile $F(z)$ satisfies Condition C. Then for all i

$$\lim_{x\to 0}\left[\theta x^{\theta-1}g_i(x^\theta)\right] = \sum_{j=1}^{n}\lambda_j > 0, \tag{21}$$

where $\theta = \max_i \theta_i$, and

$$\lim_{x\to 0}\frac{xg_i(x)}{1-G_i(x)} = \frac{1}{\theta}. \tag{22}$$

Proof: Differentiating both sides of (20) with respect to z, evaluating at $z = x^\theta$ and multiplying by $\theta x^{\theta-1}$ we obtain

$$\theta x^{\theta-1}g_i(x^\theta) = \sum_{j=1}^{n}\theta x^{\theta-1}f_j(x^\theta)\prod_{k\neq j}F_k\left(\frac{w_j\kappa_{ik}}{w_k\kappa_{ij}}x^\theta\right)$$

for all i. Taking limits as $x \to 0$ gives

$$\begin{aligned}\lim_{x\to 0}\left[\theta x^{\theta-1}g_i(x^\theta)\right] &= \lim_{x\to 0}\sum_{j=1}^{n}\theta x^{\theta-1}f_j(x^\theta)\prod_{k\neq j}F_k(a_{ijk}x^\theta)\\ &= \lim_{x\to 0}\sum_{j=1}^{n}\theta x^{\theta-1}f_j(x^\theta)\end{aligned}$$

where the second equality follows since C3 implies that $\lim_{x\to 0}F_k(a_{ijk}x^\theta) = 0$).

Now we use the change of variable $y = x^{\theta/\theta_j}$ on the term on the right, to conclude that

$$\begin{aligned}\lim_{x\to 0}\left[\theta x^{\theta-1} g_i(x^\theta)\right] &= \lim_{y\to 0}\sum_{j=1}^{n}\frac{\theta}{\theta_j} y^{1-\theta_j/\theta}\theta_j y^{\theta_j-1} f_j(y^{\theta_j}) \\ &= \lim_{y\to 0}\sum_{j=1}^{n}\frac{\theta}{\theta_j} y^{1-\theta_j/\theta}\lambda_j.\end{aligned}$$

where the second equality follows from C2. Now for those countries j such that $\theta_j = \theta$ the j-th term in the sum on the right is $\lambda_j > 0$. For the terms with $\theta_j = \theta$, the j-th term in the sum is $(\theta/\theta_j)\, y^{1-\theta_j/\theta}\lambda_j$ with converges to 0 as $y \to 0$. Thus the limit on the right equals

$$\lim_{x\to 0}\left[\theta x^{\theta-1} g_i(x^\theta)\right] = \sum_{\theta_j=\theta}\lambda_j > 0.$$

Then (22) follows Proposition 3. Since (22) implies that $\theta_j = \theta$ for all j, (21) also follows. □

We now turn to the study of the basic dynamic system, equations (19) and (20). We know that the distribution of cost $\tilde{z}$ is collapsing on zero as time passes, so it is convenient to work with the distribution of the random variable $e^{\nu t}\tilde{z}$, where ν is the common growth rate on a balanced growth path. We refer to $e^{\nu t}\tilde{z}$ as *normalized cost.* The right cdf of this normalized cost is

$$\Pr\{e^{\nu t}\tilde{z} \geq z \text{ at } t\} = \Pr\{\tilde{z} \geq e^{-\nu t} z \text{ at } t\} = F(e^{-\nu t} z, t\}.$$

On a balanced growth path the distribution of normalized cost does not vary with t so we define a balanced growth as a common growth rate ν and a profile of cost distributions $\Phi = (\Phi_1..., \Phi_n)$ such that

$$\Phi_i(z) = F_i(e^{-\nu t} z, t)$$

for all t. Along a BGP we can substituting $\Phi_i(e^{\nu t} z)$ for $F_i(z,t)$ in (19) and $\phi_i(e^{\nu t} z) e^{\nu t}$ for $f_i(z,t)$, and applying the homogeneity property in Lemma 1 of Section 3 we have that the BGP (Φ, ν) must satisfy

$$\frac{\partial \log\left(\Phi_i(e^{\nu t} z)\right)}{\partial t} = \alpha_i \log\left(\sum_{j=1}^{n}\int_{e^{\nu t} z}^{\infty}\phi_j(y)\prod_{k\neq j}\Phi_k\left(\frac{w_j\kappa_{ik}}{w_k\kappa_{ij}} y\right) dy\right).$$

Letting $x = e^{\nu t} z$, we have

$$\nu \frac{x\phi_i(x)}{\Phi_i(x)} = -\alpha_i \log \left(G_i(x)\right), \tag{23}$$

where it is understood that $G_i(x)$ is defined in term of the profile of Φ instead of F.

The next result links the growth rate of the economy with the elasticity of the distribution of sellers at zero, and establishes that the left tails of the stationary cost distributions are identical in all locations.

Proposition 7. Assume that the profile of stationary distributions $\Phi = (\Phi_1(z), ..., \Phi_n(z))$ satisfies Condition C, and let $\theta = \max_i \theta_i$. Then the growth rate on the balanced growth path equals

$$\nu = \theta \sum_{i=1}^{n} \alpha_i \tag{24}$$

and the limits

$$\lim_{x \to 0} \theta x^{\theta - 1} \phi_i(x^\theta) = \frac{1}{n} \sum_{j=1}^{n} \lambda_j \tag{25}$$

and

$$\lim_{x \to 0} \frac{x\phi_i(x)}{1 - \Phi_i(x)} = \frac{1}{\theta} \tag{26}$$

hold for all i.

Proof: Evaluating both sides of (23) at x^θ, multiplying by θ, dividing by x, and taking the limit as $x \to 0$ yields

$$\nu \lim_{x \to 0} \theta x^{\theta - 1} \phi_i(x^\theta) = -\theta \alpha_i \lim_{x \to 0} \frac{\log G_i(x^\theta)}{x}.$$

Applying l'Hospital's rule to the right side yields

$$\begin{aligned} \lim_{x \to 0} \frac{\log G_i(x^\theta)}{x} &= -\frac{1}{G_i(x^\theta)} g_i(x^\theta) \theta x^{\theta - 1} \\ &= -\sum_{j=1}^{n} \lambda_j \end{aligned}$$

using (21) and (22) in Proposition 7 and the fact that $G_i(0) = 1$. Thus

$$\nu \lim_{x \to 0} \theta x^{\theta-1} \phi_i(x^\theta) = \theta \alpha_i \sum_{j=1}^{n} \lambda_j. \tag{27}$$

Applying Proposition 7 to the left side of (10) gives

$$\nu \lambda_i = \theta \alpha_i \sum_{j=1}^{n} \lambda_j$$

and summing both sides over i gives

$$\nu = \theta \sum_{i=1}^{n} \alpha_i.$$

This verifies (24) and then (25) follows from (27). Finally, (26) follows from Proposition 3.□

As in Section 2, we are interested in conditions on the initial knowledge distributions $F_i(z, 0)$ that will imply convergence to a balanced growth path, in the sense of

$$\lim_{t \to \infty} \frac{\log[F_i(\left(e^{-(\nu/\theta)t} z\right)^\theta, t)]}{z} = -\lambda_i \tag{28}$$

for all t. Here we assume profiles for initial distributions $F_i(z, 0)$ and limiting distributions $\Phi_i(x)$ that both satisfy Condition C. Proposition 8 then implies that equations (24)-(26) hold for the common value $\theta > 0$ and positive values $\lambda_1, ..., \lambda_n$. In this n country case, $\nu = \theta \sum_{i=1}^{n} \alpha_i$. It is certainly not the case that these conditions will imply (28) for all values of z for all but the next result shows that (28) holds in the limit as $z \to 0$ and provides a characterization of the dynamics of the left-tail of the cost distributions $F_i(z, t)$.

Proposition 8: Assume that $F(z, 0)$ satisfies Condition C. Then if

$$\lambda_i(t) = -\lim_{z \to 0} \frac{\log\left[F_i((e^{-(\nu/\theta)t} z)^\theta, t)\right]}{z} \tag{29}$$

then

$$\lambda_i(t) - \lambda(0) = [\lambda_i(0) - \lambda(0)]\, e^{-(\nu/\theta)t} \tag{30}$$

and

$$\lim_{z\to 0}\frac{zF_i(z,t)}{1-F_i(z,t)}=\frac{1}{\theta}. \tag{31}$$

Proof: Decomposing the time derivative of $F_i((e^{-\alpha nt}z)^\theta,t)$ in the usual way, we have

$$\begin{aligned}\frac{dF_i(\left(e^{-(\nu/\theta)t}z\right)^\theta,t)}{dt} &= \frac{\partial F_i(\left(e^{-(\nu/\theta)t}z\right)^\theta,t)}{\partial z}\frac{d\left(e^{-(\nu/\theta)t}z\right)^\theta}{dt}+\frac{\partial F_i(\left(e^{-(\nu/\theta)t}z\right)^\theta,t)}{\partial t}\\ &= (\nu/\theta)\,f_i(\left(e^{-(\nu/\theta)t}z\right)^\theta,t)\theta\left(e^{-(\nu/\theta)t}z\right)^{\theta-1}e^{-(\nu/\theta)t}z+\frac{\partial F_i(\left(e^{-(\nu/\theta)t}z\right)^\theta,t)}{\partial t}\end{aligned}$$

and so, dividing by $F_i(\left(e^{-(\nu/\theta)t}z\right)^\theta,t)$ and applying (19) to the last term on the right,

$$\begin{aligned}\frac{d\log F_i(\left(e^{-(\nu/\theta)t}z\right)^\theta,t)}{dt} &= x\frac{(\nu/\theta)\,f_i(\left(e^{-(\nu/\theta)t}z\right)^\theta,t)\theta\left(e^{-(\nu/\theta)t}z\right)^{\theta-1}e^{-(\nu/\theta)t}z}{F_i(\left(e^{-(\nu/\theta)t}z\right)^\theta,t)}\\ &\quad+\alpha_i\log G_i(\left(e^{-(\nu/\theta)t}z\right)^\theta,t).\end{aligned}$$

Let $x=e^{-(\nu/\theta)t}z$ and divide through by x to obtain

$$\frac{d}{dt}\frac{\log F_i(x^\theta,t)}{x}=(\nu/\theta)\,\frac{f_i(x^\theta,t)\theta x^{\theta-1}x}{F_i(x^\theta,t)x}+\alpha_i\frac{\log G_i(x^\theta,t)}{x}.$$

Now let $x\to 0$

$$-\frac{d}{dt}\lim_{x\to 0}f_i(x^\theta,t)\theta x^{\theta-1}=(\nu/\theta)\lim_{x\to 0}f_i(x^\theta,t)\theta x^{\theta-1}-\alpha_i\lim_{x\to 0}g_i(x^\theta,t)\theta x^{\theta-1}.$$

Reversing signs to conform to the definition of $\lambda_i(t)$,

$$\frac{d}{dt}\lambda_i(t)=-(\nu/\theta)\,\lambda_i(t)+\alpha_i\sum_{i=1}^n\lambda_i(t).$$

Summing both sides over i we have

$$\frac{d}{dt}\sum_{i=1}^n\lambda_i(t)=-(\nu/\theta)\sum_{i=1}^n\lambda_i(t)+\sum_{i=1}^n\alpha_i\sum_{i=1}^n\lambda_i(t)=0.$$

Letting $\lambda(t) = (1/n) \sum_{i=1}^{n} \lambda_i(t)$ we have then

$$\frac{d}{dt} [\lambda_i(t) - \lambda(0)] = -(\nu/\theta) [\lambda_i(t) - \lambda(0)]$$

and (30) follows. Proposition 3 then implies (31). $\square$

The next theorem establish the existence of balanced growth path distributions.

Proposition 8.b: For any $\mathbf{w}$, $\mathbf{K}$, $\theta > 0$, and $\lambda > 0$. There is at least one non-degenerate balanced growth path $(\Phi, \nu) = (\Phi_1, ..., \Phi_n, \nu)$ that satisfies

$$\theta = \lim_{z \to 0} \frac{z\phi_i}{1 - \Phi_i} \text{ and } \lambda = \lim_{z \to 0} \theta z^{\theta - 1} \phi_i(z^\theta)$$

for all i. Moreover, any balanced growth path distributions consistent with a pair (θ, λ) are bounded between the following two Weibull distributions:

$$\exp\left(-\lambda z^{1/\theta}\right) \leq \Phi_i(z) \leq \hat{E} \exp\left(-\lambda (az)^{1/\theta}\right)$$

where $a = \min_{i,j,k} \frac{w_j \kappa_{i,k}}{w_k \kappa_{ij}}$.

Proof: We will fix K and $\mathbf{w}(t) = \mathbf{w}$, for all t, and consider the law of motion of the corresponding cost distribution. Let $F_i(z, 0)$ be any distribution with tails consistent with a pair (λ, θ). Given the diffussion process, the distribution of cost only gets lower and hence it converges. The only issue is whether the limit distribution, after normalizing it by $\exp(-\nu t)$, is degenerate. To show that it is not degenerate we bound its path by two alternative trajectories of distributions generated with law of motions to be specified below, each of them with a non-degenerate limit.

The first corresponds to $\mathbf{K} = I_{n \times n}$ and $w = 1_n$, the case of costless trade and equal wages. The path of F is higher at each t by the monotonicity of G established in Proposition 9. The limit distribution has lower cost and it is not degenerate, since it can be verified that this law of motion is identical to the one of the close economy case, and hence it converges to a Weibull with parameters λ and θ.

The second corresponds to changing the starting distribution of cost by $F_i^*(z, 0) = F_i(az, 0)$ and considering the law of motion that corresponds to the case of costless trade and equal wages. By examining G with the true initial distribution $F(., 0)$ and G with $F^*(., 0)$ it can be seen that the resulting cost distribution are stochastically higher. The limit of this law of motion is also Weibull with parameters $\lambda a^{1/\theta}$ and θ. □

Proposition 8b shows the existence of a balanced growth path for arbitrary cost parameters K and a wage vector w. Using this result together with Proposition 5 of Section 3, we can show that for any K and L there is a balanced growth cost distribution, and associated wages w which displays balanced trade.

4.1 The case of n symmetric countries

This section analyzes the case of n symmetric countries. First we show that on a balanced growth path with costless trade, all countries share the same stationary distribution of cost, which is of the Weibull family. The corresponding stationary distribution of productivity is Frechet. Therefore this case provides a benchmark, closely related to the distribution of productivities used in the trade theory of Eaton and Kortum (2002) and Alvarez and Lucas (2007). This benchmark is also of interest as it provides a lower bound on the distribution of cost in any balanced growth path.

The following result shows that all G_i are bounded from below by the joint distribution of sellers that would be active in i in a hypothetical world economy with no trade costs and a common labor market.

Proposition 9: $G_i(z;\mathbf{K},\mathbf{w}) \geq \prod_{j=1}^{n} F_j(z)$, with equality if $\mathbf{K}=\mathbf{I}$ and $\mathbf{w}=\mathbf{1}$.

Proof: Define the sets

$$M(z) = \left\{\mathbf{z} \in \mathbb{R}^n_+ : \min\{z_1,...,z_n\} \geq z\right\}$$

and

$$B_i(z;\mathbf{w},\mathbf{K}) = \left\{\mathbf{z} \in \mathbb{R}^n_+ : z_{j^*} \geq z, \text{ where } j^* = \arg\min_{j\in\{1,\ldots,n\}} \left\{\frac{w_j z_j}{k_{ij}}\right\}\right\}.$$

It is easy to see that $M(z) \subseteq B_i(z;\mathbf{w},\mathbf{K})$ since $\min\{z_1,...,z_n\} \geq z \Rightarrow z_j \geq z$, all $j=1,...,n$. Therefore,

$$G_i(z;\mathbf{K},\mathbf{w}) = \int_{\mathbf{z}\in B_i(z;\mathbf{w},\mathbf{K})} f(\mathbf{z},t)d\mathbf{z} \geq \int_{\mathbf{z}\in M(z)} f(\mathbf{z},t)d\mathbf{z} = \prod_{j=1}^{n} F_j(z,t).\square$$

The balanced growth path of a symmetric world with costless trade is characterized by

Proposition 10: Assume that the n countries have the same size, $L_i = L$, and the same $\alpha = \alpha_i$, and that trade is costless, $\kappa_{ij} = 1$, all i, j. Then the world economy is on a balanced growth path if and only if wages are the same, $w_i = 1$ all i, the right CDFs are given by

$$\Phi_i(z) = \exp(-\lambda z^{\frac{1}{\theta}})$$

for some $\lambda > 0$ and $\theta > 0$, and the growth rate of each of the economies is

$$\nu = n\alpha\theta.$$

Proof: The distribution of sellers varies across countries only through its dependence on country specific trade costs (see Equation 20). Therefore, in the case of costless trade all countries share the same distribution of sellers, $G_i(z) = G(z)$. In this case, the stationary distribution of cost for every country i solves

$$\frac{\partial \log(F_i(z,t))}{\partial t} = \alpha \log(G(z,t)). \tag{32}$$

In this symmetric case,

$$\begin{aligned} G(z,t) &= \int_z^\infty n f(y,t) \left[F(y,t)\right]^{n-1} dy \\ &= F(z,t)^n \end{aligned}$$

Iso we can drop the subscripts and write

$$\frac{\partial \log(F(z,t))}{\partial t} = \alpha n \log(F(z,t)) \tag{33}$$

for the common right cdf F. Along a BGP we can replace $F(z,t)$ with $\Phi(e^{\nu t}z)$ and let $x = e^{\nu t}z$ to obtain

$$\frac{\phi(x)x}{\Phi(x)} = -\frac{\alpha n}{\nu} \log(\Phi(x))$$

The desired result then follows from applying Proposition 1 to this equation after replacing α with $n\alpha$. $\square$

If we start from a situation with costless trade and equal wages, a marginal increase in trade cost or wages has a negligible effect in the distribution of sellers.

Proposition 11:

$$\left.\frac{\partial G_i(z;\mathbf{K},\mathbf{w})}{\partial \kappa_{ij}}\right|_{\mathbf{K}=\mathbf{I},\mathbf{w}=\mathbf{1}} = \left.\frac{\partial G_i(z;\mathbf{K},\mathbf{w})}{\partial w_j}\right|_{\mathbf{K}=\mathbf{I},\mathbf{w}=\mathbf{1}} = 0.$$

Proof: Differentiating (2) with respect to κ_{ij}

$$\frac{\partial G_i(z;\mathbf{K},\mathbf{w})}{\partial \kappa_{ij}} = \int_z^\infty \left[f_j(y) \sum_{k\neq j} \left(-\frac{w_j\kappa_{ik}}{w_k}\frac{1}{\kappa_{ij}^2}y\right) f_k\left(\frac{w_j\kappa_{ik}}{w_k\kappa_{ij}}y\right) \prod_{l\neq j,k} F_l\left(\frac{w_j\kappa_{il}}{w_l\kappa_{ij}}y\right) + \sum_{k\neq j} f_k(y)\frac{w_k}{w_j\kappa_{ik}} y f_j\left(\frac{w_k\kappa_{ij}}{w_j\kappa_{ik}}y\right) \prod_{l\neq j,k} F_l\left(\frac{w_l\kappa_{ik}}{w_k\kappa_{il}}y\right)\right] dy.$$

Evaluating at $\mathbf{K}=\mathbf{I}$ and $\mathbf{w}=\mathbf{1}$

$$\begin{aligned}
\left.\frac{\partial G_i(z;\mathbf{K},\mathbf{w})}{\partial \kappa_{ij}}\right|_{\mathbf{K}=\mathbf{I},\mathbf{w}=\mathbf{1}} &= \int_z^\infty \left[f_j(y)\sum_{k\neq j}(-y)\, f_k(y) \prod_{l\neq j,k} F_l(y) + \sum_{k\neq j} f_k(y) y f_j(y) \prod_{l\neq j,k} F_l(y)\right] dy \\
&= \int_z^\infty \left[-y f_j(y) \sum_{k\neq j} f_k(y) \prod_{l\neq j,k} F_l(y) + y f_j(y) \sum_{k\neq j} f_k(y) \prod_{l\neq j,k} F_l(y)\right] dy \\
&= 0.
\end{aligned}$$

Similarly, differentiating (2) with respect to w_j

$$\frac{\partial G_i(z;\mathbf{K},\mathbf{w})}{\partial w_j} = \int_z^\infty \left[f_j(y) \sum_{k\neq j} \frac{\kappa_{ik}}{w_k\kappa_{ij}} y f_k\left(\frac{w_j\kappa_{ik}}{w_k\kappa_{ij}}y\right) \prod_{l\neq j,k} F_l\left(\frac{w_j\kappa_{il}}{w_l\kappa_{ij}}y\right) + \sum_{k\neq j} f_k(y)\left(-\frac{w_k\kappa_{ij}}{\kappa_{ik}}\frac{1}{w_j^2}y\right) f_j\left(\frac{w_k\kappa_{ij}}{w_j\kappa_{ik}}y\right) \prod_{l\neq j,k} F_l\left(\frac{w_l\kappa_{ik}}{w_k\kappa_{il}}y\right)\right] dy.$$

Evaluating at $\mathbf{K} = \mathbf{I}$ and $\mathbf{w} = \mathbf{1}$ and rearranging terms

$$\begin{aligned}\left.\frac{\partial G_i(z;\mathbf{K},\mathbf{w})}{\partial w_j}\right|_{\mathbf{K}=\mathbf{I},\mathbf{w}=\mathbf{1}} &= \int_z^{\infty}\left[-yf_j(y)\sum_{k\neq j} f_k\left(y\right)\prod_{l\neq j,k} F_l\left(y\right)\right.\\ &\qquad \left.+yf_j(y)\sum_{k\neq j} f_k(y)\prod_{l\neq j,k} F_l(y)\right]dy\\ &= 0.\end{aligned}$$

□

Proposition 11 holds independently of the profile of $F(z)$ of right cdfs, but it takes as given the profile of wages $\mathbf{w}$ which we know is determined by the profile $F(z)$. The following result establishes that when starting from a world with symmetric countries and costless trade, changes in trade cost or in the size of an individual country have a negligible effect on the profile of stationary distributions.

Let the parameters of a world economy be given by n, α, $\mathbf{K}$ and $\mathbf{L}$. We are interested in the comparative static of the profile of stationary distributions $\phi(x;\mathbf{K},\mathbf{L})$ with respect to $\mathbf{K}$ and $\mathbf{L}$.

Mapping H: For fixed $\theta > 0$ and $\lambda > 0$, the profile $\phi(x;\mathbf{K},\mathbf{L})$ is the solution to the functional equation $H(\phi,\mathbf{K},\mathbf{L})$ given by: equation (23) defining a balance growth path, equation (20) defining the distribution of sellers, equation (17) giving the by the solution to the static trade equilibrium, $\nu = n\alpha\theta$ defining the growth rate of a balance growth path, and $\lim_{x\to 0}\frac{x\phi_i(x)}{\Phi_i(x)} = \theta$ and $\lim_{x\to 0}\theta x^{\theta-1}\phi_i(x^{\theta}) = \lambda$ giving the boundary conditions for the densities.

The reason why θ and λ need to be given is that there are multiple steady state, but these two parameters determine the basin of attraction, as discussed in Proposition 8. In addition, note that for $\mathbf{K} = \mathbf{I}$ and $\mathbf{L} = \mathbf{1}$ the solution of (6) is given by Proposition 10.

Proposition 12: Let ϕ be the solution to $H(\phi,\mathbf{K},\mathbf{L}) = 0$. Assume that $H(\cdot,\mathbf{K},\mathbf{L})$ is differentiable, and that it is invertible. Then

$$\left.\frac{\partial\phi_i(z;\mathbf{K},\mathbf{L})}{\partial\kappa_{ij}}\right|_{\mathbf{K}=\mathbf{I},\mathbf{L}=\mathbf{1}} = \left.\frac{\partial\phi_i(z;\mathbf{K},\mathbf{L})}{\partial L_j}\right|_{\mathbf{K}=\mathbf{I},\mathbf{L}=\mathbf{1}} = 0.$$

Proof: The result follows from totally differentiating $H(\phi, \mathbf{K}, \mathbf{L}) = 0$ and Proposition 11.

Remark E'. Before finishing this section we note specialize our remark on the elasticity of substitution η for the case of n symmetrical countries. Since in this case equilibrium wages are the same for all countries and constant through time, the path of the distribution of cost is independent of the elasticity of substitution η. While the distribution of cost is independent of η, the value of real gdp and of trade to gdp, the volume of trade, both depend on η.

4.2 From Locations to Countries

In order to clarify the role of scale effects and the interpretation of countries of different sizes we introduce the notion of a location within a country. We consider a world economy consisting of n countries, where each country i contains m_i locations. To simplify the analysis, we assume that a country is defined by a set of locations satisfying the following conditions:[6]

- within each country there is a common labor market across the m_i locations;
- there are no trading cost between locations within a country;
- locations within a country face the same trading cost when trading with locations in other countries.

Denoting by $L_{i,l}$ the labor force in location l of country i, we can write the labor force of country i as

$$L_i = \sum_{l=1}^{m_i} L_{i,l},$$

[6] It is straightforward to extend the analysis to the case where labor is not mobile across location and there are arbitrary transportation costs across locations. In this case an equilibrium is given by a wage vector of dimension $\sum_{i=1}^{n} m_i$.

Denoting by $F_{i,l}(z,t)$ the distribution of best practices in location l of country i, we can write the distribution of best practices in country i as

$$F_i(z,t) = \prod_{l=1}^{m_i} F_{i,l}(z,t).$$

Notice that since there is no transportation cost between location within a country, and that all locations within a country share the same transportation cost vis-a-vis all other countries, the distribution $F_i(z,t)$ of best practices of a country is all we need to know to calculate a static trade equilibrium. In particular, given the distribution of best practices in each country and the size of the labor force of each country, L_i, we can calculate a static trade equilibrium as described in Section 3.

Finally, assuming that the arrival rate of ideas in location l of country i equals $\alpha_{i,l}$, we can aggregate the evolution of best practices of all locations in a country to obtain the law of motion of best practices in country i:

$$\sum_{l=1}^{m_i} \frac{\partial \log\left(F_{i,l}(z,t)\right)}{\partial t} = \sum_{l=1}^{m_i} \alpha_{i,l} \log\left[G_i(z,t)\right]$$

or

$$\frac{\partial \log\left(F_i(z,t)\right)}{\partial t} = \alpha_i \log\left[G_i(z,t)\right]$$

where $\alpha_i = \sum_{l=1}^{m_i} \alpha_{i,l}$.

Furthermore, assuming that countries are aggregates of different different number of symmetric locations in terms of their population and number of technology managers, $L_{i,l} = L$ and $\alpha_{i,l} = \alpha$, we have that countries of different size are obtained by scaling their population $L_i = m_i L$ and the arrival rate of ideas $\alpha_i = m_i \alpha$.

It should be also clear that, provided that the structure of transportation cost and labor markets across locations is kept constant, an equilibrium of the model is invariant to arbitrary splits of locations into countries. For instance, a country with m_i locations can be spliced into m_i individual countries, each of them with a population of size L, receiving α ideas per period, and having a distribution of best practices $F_{i,l}(z,t) = \left(F_i(z,t)\right)^{\frac{1}{m_i}}$.

5 Quantitative Exploration

In this section we present preliminary numerical examples to illustrate the effect of trade costs on the stationary distribution of productivity, per-capita income, trade patterns, and the transitional dynamics of these variables following a decline in trade costs. We consider two cases: (1) a world consisting of n symmetric countries facing trade costs $\kappa_{ij} = \kappa$, and (2) world consisting of 1 asymmetric and $n-1$ symmetric locations facing trade costs $\kappa_1 = \kappa_{1j} \leq \kappa_{ji} = 1$, $j = 2, ..., n$, $i \neq j$.

5.1 Calibration

We can gain some understanding of the order of magnitude of the parameters α and θ by using information of the long-run growth rate of the economy ν, and information on θ which instead can be obtained either from the dispersion of productivities, or the tail of the size distribution of firms/plants, or the magnitude of trade elasticities. We turn to the description of each of these approaches.

First, since we show above that asymptotically z is Weibull distributed, then $\log(1/z)$, the log of productivity, has standard deviation equal to $\theta\pi/\sqrt{6}$, see chapter 3.3.4 of Rinne (2008). Hence we can take measures of dispersion of (log) productivity to calibrate θ. The dispersion of (log) productivity range from $0.6 - 0.75$ when measured as the value-added per worker – see Bernard et al. (2003) Table II – and around 0.8 when measures of physical total factor productivity are obtained using data on value-added, capital and labor inputs, and assumptions about the demand elasticities – see Hsieh and Klenow (2009) Table I, dispersion of TFPQ.[7] These numbers suggest a value for θ in the range $[0.5, 0.6]$.

Second, using that productivity is asymptotically distributed Frechet, and that the tail of the Frechet behaves as that of a Pareto distribution with tail coefficient $1/\theta$, we can use data on the tail of the distribution of productivity to directly infer θ. Lacking direct information on physical productivities, we can use information on the tail of the distribution of employment, together

[7] Using data for eleven products for which direct measures of physical output are available Haltiwanger et al. (2008) calculate true measures of physical total factor productivity. They find that the dispersion of (log) true physical productivity is 20% higher than that measured using just value-added information.

with a value for the elasticity of demand.[8] The tail of the size (employment) distribution of firms is approximately equal to 1.06 – see Figure 1 in Luttmer (2007). Therefore for the values of demand elasticities typically considered in the literature, say $\eta \in [3, 10]$, (see Broda and Weinstein (2006), Imbs and Mejean (2010), or Hendel and Nevo (2006)) it would imply a value for θ in the range $[0.1, 0.5]$.

Third, as showed before, in the case of a model with several symmetrical locations, θ is approximately the Armington trade elasticity, which will also give us another way to measure it. This method would suggest a value for θ in $[0.1, 0.25]$ (Alvarez and Lucas, 2007).

Once we have an estimate of θ, whatever its source, together with an estimate of long term growth of output ν, we can estimate the value of α, using that $\nu = \alpha\theta$. For instance, if we take the long-run growth to be 0.02, α would be in the range $[0.03, 0.2]$. Note that with a value of $\alpha = 0.1$, the half-life to convergence will be approximately 5 years.

Based on this discussion, we set $\theta = 0.2$ to be consistent with the available evidence on the right tail of the distribution of productivity, and set $\alpha = 0.02\theta n$, to match a growth rate 0.02. We consider a world consisting of $n = 50$ economies symmetric in all dimensions, with the possible exception of their trade cost. Given our choice of n, in a world with symmetric trade cost each economy has a relative GDP similar to that of Canada or South Korea.

5.2 Symmetric World

Figure 1 illustrates the long run effect on the distribution of productivities $1/z$ of introducing trade costs in a symmetric word of n countries. The thought experiment is to go from costless trade to a case where κ_{ij} takes a common value $\kappa < 1$ for all $j \neq i$.) On the x-axis we measure the value of productivity, expressed as a ratio to the average productivity in the economy with costless trade ($\kappa = 1$). On the y-axis we display the density of relative productivities for different values of κ. The top panel shows the densities of productivities of the potential producers, a transformation of the common density of the cost ϕ (or of f). The bottom panel shows the density of productivities of the sellers active in each country, a transformation of the

[8]The CES structure implies that employment at industry/firm with cost x satisfies $l(z) \propto (1/z)^{\eta-1}$.

common density g.

Note first that, due to the selection due of trade, the density of sellers is stochastically larger than the one of potential producers, for each κ. The difference between the two densities increases for larger trade cost (for lower values of κ). Second, note that for $\kappa = 1$ the densities are Frechet, as we showed in Proposition 10. Third, for larger trade cost (lower κ) both densities have a thicker left tail, especially so for potential producers. Fourth, the change in the distribution of potential producers as κ varies illustrates the effect of trade costs on the diffusion of technologies, the main feature of the model in this paper. Finally, we remind the reader that, as stated in Remark E', these distributions are independent of the value of η.

Figure 2 illustrates the effect of introducing symmetric trade costs in real gdp C in the top panel and in the trade volume (the ratio of imports to gdp) in the bottom panel in a symmetric word of n countries.[9] On the x-axis we measure trade cost. On the y-axis we measure real gdp, relative to gdp under costless trade (top panel) or the trade share, relative to the costless trade benchmark (bottom panel). In each panel the solid line displays the impact effect of introducing the trade costs, calculated by holding the distribution of productivity fixed at its distribution under of costless trade. As shown above, this benchmark has a Frechet distribution. The other lines in each panel show the effect on the balanced growth path of introducing trade cost. Each line is for a different value of the elasticity of substitution η. Recall that the growth rate of the world economy is unaffected by the introduction of finite trade cost, as long as $\kappa > 0$, so the ratios on these panels should be interpreted as level effects around the common balanced growth path.

The solid lines showing the impact effects correspond to the familiar effects of trade cost in the Ricardian trade theory of Eaton and Kortum (2002) and Alvarez and Lucas (2007). In this case there is an analytical expression

[9] The value of imports to country j from i is given by:

$$I_{ji} = \int_{\mathbf{B}_{ji}} p_j(\mathbf{z})c_j(\mathbf{z})f(\mathbf{z})d\mathbf{z} = \int_0^\infty \left(\frac{w_i\, z_i}{\kappa_{ji}\, p_j}\right)^{1-\eta} w_j L_j\, f_i(z_i) \prod_{k\neq i} F_k\left(\frac{w_i\kappa_{jk}}{w_k\kappa_{ji}}\, z_i\right) dz_i \ ,$$

so total imports in country j are $I_j = \sum_{i=1,i\neq j}^{n} I_{ji}$ and volume of trade, defined as imports relative to GDP, is given by $v_j = 1/(1 + I_{jj}/I_j)$.

for the GDP of an economy relative to case of costless trade:[10]

$$\frac{C(\kappa)}{C(1)} = \frac{\left[1 + (n-1)\kappa^{\frac{1}{\theta}}\right]^{\theta}}{n^{\theta}}.$$

The output effects of trade cost depend only on θ, the country size, $1/n$, and the value of the trade cost κ. As it has been noted, this expression does not depend on the value of the substitution elasticity η. In contrast, once trade costs the value of η does matter, as showed by the dashed lines. These effects of trade costs on gdp are larger the more difficult it is to substitute domestic goods for imports. The long-run calculations include the effects of the changes in the distribution of productivity due to the diffusion of technology, which are the contribution of this paper. As trade cost increases, individuals in each country meets relatively more worst sellers, and therefore, the good technologies diffuse more slowly.

This panel shows that the difference between the effect on impact (solid line) and the long-run effects (any of the other lines) are extremely small in the neighborhood of costless trade. This is to be expected, since Proposition 12 shows that around the symmetric costless trade, trade cost have only second order effects on productivity. Indeed, Figure 1 shows that the lesson drawn from Proposition 12 applies for a large range of trade costs, say eve n trade cost as large as $\kappa \geq 0.5$.

The effect of trade cost on the volume of trade is shown in the bottom panel of Figure 2. The impact effect and long-runs effects are defined as in the top panel. Note that the impact effect of trade is the same as in Alvarez and Lucas (2007), since the distribution of productivities is Frechet, and it is given by[11]

$$v = \frac{(n-1)\kappa^{1/\theta}}{1 + (n-1)\kappa^{1/\theta}} \tag{34}$$

The long run effect of trade cost on the volume of trade is smaller than its effect on impact. This is due to the fact that a higher trade cost leads to a distribution of productivity for potential producers with a thicker left tail and the same right tail (see Figure 1), i.e. they lead to larger dispersion of productivities. A larger dispersion of productivity is associated with larger

[10]This formula follows from specializing equation (6.10) in Alvarez and Lucas (2007) to a world without intermediate goods, non-tradable goods, and tariffs.

[11]See Alvarez and Lucas (2007) expression (6.11) for the case of $\beta = \omega = 1$ and $\alpha = 0$.

gains from trade. As discussed in Remark E', this distribution of productivities are independent of the value of η. But the gains from trade are not independent of η, and with larger elasticities of substitution for any given κ there is more trade.

5.3 Asymmetric World and Catch-up Growth

In the previous exercises we illustrated the effect of symmetric trade barriers. We now explore the impact of unilateral trade barriers by considering a world economy consisting of n countries, $n-1$ of which face symmetric trade cost κ_1 when trading among themselves, and a single country that faces a relatively larger cost to trade from and to this country, $\kappa_n \leq \kappa_1$. We refer to the first group as the $n-1$ symmetric countries, and to the later as the small open economy. We interpret the $n-1$ relatively open countries as developed economies. Following Alvarez and Lucas (2007), we calibrate their trade cost to $\kappa_1 = 0.75$.[12]

In Figure 3 we illustrate how the balanced growth path of these economies is affected by the cost of trade with the small open economy, κ_n. In the top panel we show the effect on the per-capita income of the $n-1$ symmetric countries (solid line) and the small open economy (dashed line). Similarly, in the bottom panel we show the effect on the volume of trade. Most of the impact occurs in the small open economy, with the effect being more pronounced than those reported in Figure 1 for the case of n symmetric countries.

Figure 4 displays the dynamic effects of a once-and-for-all trade liberalization in the small economy, taking the form of a reduction of its trade costs to the level of the advanced economies. These dynamics are shown for the three initial (pre-liberalizatiuon) trade costs $\kappa_n(0) = 0.05, 0.30, 0.50$.

The main message from Figure 4 is that a large part of the output gains from a reduction in trade costs occur immediately. The distribution of productivity of the small open economy is not affected on impact, but this economy is no longer forced to rely on its own producers for most of the goods it consumed, and can therefore discontinue its most unproductive technologies. In the model, this effect happens immediately. Thereafter, the distribution of productivity continues to improve, but this effect tends to be less important:

[12] This value is a compromise between low values of κ obtained from indirect estimates using gravity equations and higher ones using direct evidence of transportation costs, e.g., freight charges, imputed time costs on cargo in transit.

As Figure 1 illustrates, the right tail of productivity is less affected by trade barriers.

6 Conclusions

We have proposed and studied a new theory of cross-country technology diffusion, constructed by integrating two existing models: a static model of international trade based on the Ricardian framework introduced by Eaton and Kortum (2002) and a stochastic-process model of knowledge growth introduced in Kortum (1997) in which individuals get new ideas through their interactions with others. The new feature that connects these two models is a *selection effect* of international trade: Trade directly affects productivity levels by replacing inefficient domestic producers with more efficient foreigners and so increasing every country's contacts with best-practice technologies around the world.

The theory implies a long-run equilibrium in which all economies share a common, constant, endogenously determined growth rate, provided they are all connected in some degree through trade. Differences in trade cost will induce differences in income *levels* but not, in the long run, in rates of growth. This feature is shared with the von Neumann (1927) model and with Parente and Prescott (1994) model of "barriers to riches." The transition dynamics following changes in trade costs, both world wide and by an individual country, are illustrated through stylized numerical examples. These dynamics are a mixture of static gains from trade that occur instantaneously under the trade model we use and gradual change that results from to changes in the intellectual environment that trade brings to individual countries. Improvements in technology arise from interactions among people who are brought together by the prospects of gains from trade and who get new ideas by adapting better technologies currently used in other locations and/or in the production of other goods.

The model of this paper is general enough to support a fairly realistic calibration to the world economy (as in Alvarez and Lucas (2007)) but our numerical illustrations here should not be viewed as an attempt to do this. The trade shares in the figures are much larger than those we observe. Adding a non-tradeables sector would remedy this, and would also reduce the size of the jump in production that follows a trade liberalization, but we have not done this. The model of technological change that we have adopted from Kortum (1997) is one of many possibilities—see, for example, the ones explored in Lucas and Moll (2011)—and we have not yet sought a parameterization that matches up to observations on actual catch-up growth. These are but two of the many directions that would be interesting to pursue further.

A Additional Results: Closed Economy

A.1 Conditions for Bounded Consumption

We discuss a conditions so that the economy does not have unbounded gdp. The expression for the price index $p(t)$ is strictly positive, or equivalently real gdp $y(t)$ is finite only if the left tail of the distribution of the cost z goes to zero at a fast enough rate relative to the elasticity of substitution η. Economically, if the goods are too close of substitutes and the distribution of cost near zero has too thick of a tail, then an economy can concentrate in the production on these goods, and obtain any amount of the final good that it desires, or equivalently $p = 0$. In what follow we will impose conditions on η and the tail of $F(\cdot, t)$ so this does not occur. A sufficient condition that is useful in our set-up, is that the elasticity of the c.d.f. of z evaluated at $z = 0$, be larger than $\eta - 1$. This is the same condition as in Eaton and Kortum (2002), but our requirement is only in a neighborhood of $z = 0$.

Lemma A.1: Let $\epsilon(t) > 0$ be the elasticity of the cdf $1 - F(z, t)$ evaluated at $z = 0$. If $\epsilon(t) > \eta - 1$, then $p(t) > 0$ and $y(t) < \infty$.

proof: To simplify the notation of the proof let $\epsilon(t) = \epsilon$. The expression for $p(t)$ contains the integral of $z^{1-\eta}\ f(z, t)$ for $z \in (0, \infty)$. If $0 \leq \eta \leq 1$ then $z^{1-\eta}$ is increasing, so the integral is strictly positive provided that $f(z, t) > 0$ for $z > 0$. When $\eta > 1$, $p(t)$ is strictly positive iff the integral of $z^{1-\eta} f(z, t)$ is finite. The function $z^{1-\eta}$ is decreasing so, the integral of this function converges if and only if the integral between 0 and $b > 0$ converges for some $b > 0$. Consider first the cdf $1 - \bar{F}(z, t) = A\ z^{\epsilon}$ in the interval $[0, b]$ for some constant $A > 0$, so that elasticity in $(0, b]$ is constant and given by ϵ. Direct computation gives $\int_a^b z^{1-\eta} z^{\epsilon - 1} dz = (1/(1 - \eta + \epsilon))\ z^{1-\eta+\epsilon}|_a^b$. This integral is finite when $\eta > 1$, provided that the limit as a goes to zero is finite. For this we require $\epsilon > \eta - 1$. Lemma A.1 shows that the ratio of two CDFs with different elasticities at zero diverges as z goes to zero. From this lemma we know that if the integral of $z^{1-\eta}$ converges when integrating with the cdf $1 - \bar{F}(z, t)$, then it will also converge using the cdf $1 - F(z, t)$. $\square$

A.2 Interpretation of θ

In Section 2 we show that there are two simple statistic that determine when an initial distribution $G(z)$ belong to the basin of attraction of a balance growth: (i) the elasticity at zero of the cdf,

$$\frac{1}{\theta} = \lim_{z\to 0} \epsilon(z) = \lim_{z\to 0} \frac{zg(z)}{1-G(z)},$$

and (ii) the density at zero of the transformation of the cost $x^{1/\theta}$,

$$\lambda = \lim_{x\to 0} \theta x^{\theta-1} g(x^\theta).$$

These two parameters, together with arrival rate of meetings, α, fully characterize the growth rate and the stationary distribution of cost in a balance growth path. Of the two parameters characterizing the initial distribution, θ is asymptotically more important as it governs the growth rate. The following Lemma, which extensively used in the proofs of various results in the paper, provides an interpretation of θ as a measure of the relative concentration of arbitrarily low costs in two distributions.

Lemma A.2: Let $1 - F_1(z)$ and $1 - F_2(z)$ be two cdf with continuous and strictly positive elasticity at 0, i.e., $\lim_{z\to 0} \epsilon_i(z) = \epsilon_i(0) > 0$, and $\epsilon_1(0) < \epsilon_2(0)$ $(\theta_1 > \theta_2)$. Then

$$\lim_{z\to 0} \frac{1-F_2(z)}{1-F_1(z)} = \lim_{z\to 0} \frac{f_2(z)}{f_1(z)} = 0.$$

Proof: Using the definition of the elasticity we can write the cdf and density functions as

$$1 - F_i(z) = [1 - F_i(\bar{z})] \exp\left[-\int_z^{\bar{z}} \frac{\epsilon_i(y)}{y} dy\right].$$

and

$$f_i(z) = [1 - F_i(\bar{z})] \frac{\epsilon_i(z)}{z} \exp\left[-\int_z^{\bar{z}} \frac{\epsilon_i(y)}{y} dy\right].$$

Therefore, we can express the ratio between the first and second density functions as

$$\frac{f_2(z)}{f_1(z)} = \frac{1-F_2(\bar{z})}{1-F_1(z)}\frac{\epsilon_2(z)}{\epsilon_1(z)} \exp\left[-\int_z^{\bar{z}} \frac{\epsilon_2(y)-\epsilon_1(y)}{y} dy\right].$$

From the continuity of $\epsilon_i(z)$ and the fact that $\epsilon_2(0)-\epsilon_1(0) > 0$ we know that for $\bar{z}$ close to zero there exist ε, $0 < \varepsilon \leq \epsilon_2(0)-\epsilon_1(0)$, such that $\epsilon_2(z)-\epsilon_1(z) \geq \varepsilon$ for all $0 < z \leq \bar{z}$. Therefore, for all $z \leq \bar{z}$

$$\begin{aligned}\frac{f_2(z)}{f_1(z)} &\leq \frac{1-F_2(\bar{z})}{1-F_1(z)}\frac{\epsilon_2(z)}{\epsilon_1(z)} \exp\left[-\int_z^{\bar{z}} \frac{\varepsilon}{y} dy\right] \\ &= \frac{1-F_2(\bar{z})}{1-F_1(z)}\frac{\epsilon_2(z)}{\epsilon_1(z)} \left(\frac{z}{\bar{z}}\right)^{\varepsilon},\end{aligned}$$

and since $f_i(z) \geq 0$, $i = 1, 2$,

$$\lim_{z\to 0} \frac{f_2(z)}{f_1(z)} = 0.$$

By l'Hopital rule, this also implies that

$$\lim_{z\to 0} \frac{1-F_2(z)}{1-F_1(z)} = 0.$$

□

A.3 Partial Converse to Proposition 3

Can the conditions for an initial distribution to belong to the basin of attraction of a balance growth path be express solely in terms of the behavior of the elasticity around zero? We restrict the set of initial conditions so that if they have a bounded elasticity then they satisfy (11). Consider the class of initial c.d.f. whose elasticity on the neighborhood of zero can be written as a sum of power functions, i.e.,

$$\epsilon(z) \equiv \frac{g(z)z}{1-G(z)} = e_0 + \sum_{i=1}^{\infty} e_i \; z^{\xi_i} + o(z) \tag{35}$$

where $\xi_i > 0$ and $\lim_{z \downarrow 0} \frac{o(z)}{z} \to 0$.[13]

The follow result provides a partial converse to Proposition 4, as it provides a sufficient condition for a initial distribution with strictly positive and finite elasticity to converges to a balance growth path as defined in (10).

Proposition A.3 Assume that the initial c.d.f. $G(z)$ has elasticity of the form given by (35), with $\sum_{i=1}^{\infty} |\frac{e_i}{\xi_i}| = A < \infty$. Then, $G(z)$ satisfies condition (11).

Proof. Definite $H(x) = G(x^\theta)$, where $\theta = 1/e_0$. Let $0 \leq x < \bar{x} < 1$ closed enough to 0. Integrating the equation defining the elasticity of $H(x)$ between x and $\bar{x}$ we obtain

$$\begin{aligned} H(x) &= H(\bar{x}) \exp\left[-\int_x^{\bar{x}} \frac{\epsilon(t)/\epsilon(0)}{t} dt\right] \\ &= H(\bar{x}) \exp\left[-\int_x^{\bar{x}} \frac{1 + \frac{1}{\epsilon(0)}\left[\sum_{i=1}^{\infty} e_i \ t^{\xi_i} + o(t)\right]}{t} dt\right] \\ &= H(\bar{x}) \frac{x}{\bar{x}} \exp\left[-\frac{1}{\epsilon(0)} \int_x^{\bar{x}} \sum_{i=1}^{\infty} e_i \ t^{\xi_i - 1} dt - \frac{1}{\epsilon(0)} \int_x^{\bar{x}} \frac{o(t)}{t} dt\right]. \end{aligned}$$

where the second equality follows from the definition of a regular elasticity. Rearrenging,

$$\begin{aligned} \frac{H(x)}{x} &= \frac{H(\bar{x})}{\bar{x}} \exp\left[-\frac{1}{\epsilon(0)} \int_x^{\bar{x}} \sum_{i=1}^{\infty} e_i \ t^{\xi_i - 1} dt\right] \exp\left[-\frac{1}{\epsilon(0)} \int_x^{\bar{x}} \frac{o(t)}{t} dt\right] \\ &= \frac{H(\bar{x})}{\bar{x}} \exp\left[-\frac{1}{\epsilon(0)} \left[\int_x^{\bar{x}} \sum_{i \in pos} e_i \ t^{\xi_i - 1} dt - \int_x^{\bar{x}} \sum_{i \in neg} |e_i| \ t^{\xi_i - 1} dt\right]\right] \\ &\quad \exp\left[-\frac{1}{\epsilon(0)} \int_x^{\bar{x}} \frac{o(t)}{t} dt\right] \end{aligned} \tag{36}$$

where $pos = \{i : e_i \geq 0\}$ and $neg = \{i : e_i < 0\}$. We notice that each of the integrals inside the first exponential increase when we lower x, as we are integrating a positive function over a larger range. Moreover, these two

[13]This class includes the cases in which $\epsilon(z)$ is differentiable at 0 (set $\xi_i \geq 1$ for all i), as well as many other cases where it is not differentiable, such as $\epsilon(z) = e_0 + \sqrt{z}$.

integrals are bounded, i.e.,

$$\begin{aligned}\int_x^{\bar{x}} \sum_{i\in pos} e_i\, t^{\xi_i-1}dt &\leq \int_x^{\bar{x}} \sum_{i=1}^{\infty} |e_i|\, t^{\xi_i-1}dt \\ &\leq \liminf_{n\to\infty} \int_x^{\bar{x}} \sum_{i=1}^{n} |e_i|\, t^{\xi_i-1}dt \\ &= \liminf_{n\to\infty} \sum_{i=1}^{n} |\frac{e_i}{\xi_i}| \left[\bar{x}^{\xi_i} - x^{\xi_i}\right] \\ &\leq \sum_{i=1}^{\infty} |\frac{e_i}{\xi_i}| = A < \infty\end{aligned}$$

where the first inequality follows from Fatou's Lemma, the second equality follows from integration, and the last inequality follows from the definition of a regular elasticity and the fact that $0 < x < \bar{x} \leq 1$. Similar arguments can be applid to show that the second integral inside of the first exponential is bounded. Finally, the absolute value of the argument of the second exponential is uniformly bounded for $\bar{x}$ small enough, i.e., $|\int_0^{\bar{x}} \frac{o(t)}{t}dt| \leq B < \infty$. Since all the integrals in the right hand side of (36) are monotone and bounded, they converge to a finite limit. This proves that $0 < \lim_{x\to 0} H'(x) = \lim_{x\to 0} H(x)/x = \lim_{x\to 0} \theta x^{\theta-1} g(x^{\theta}) < \infty$, which is equivalent to (11) as Proposition 3 shows. □

A.4 Alternative Definition of a Balance Growth Path

For completeness, we characterize the asymptotic behavior of initial distributions that do not satisfy the conditions of Proposition A.3. While these initial distributions do not converge to a balance growth path as define in (10), they converge to a balance growth path in a weaker sense as stated in the following proposition.

Proposition A.4 Assume that the initial cdf $1-G(z)$ has an elasticity that is continuous, strictly positive, and finite at zero equal $1/\theta$. Let the $q(t)$ be the q^{th} quantile of the distribution $F(z,t)$, i.e.,

$$F(d(t),t) = \exp\left[\log\left[G\left(q(t)\right)\right] e^{\alpha t}\right] = 1 - q. \tag{37}$$

Then, the distribution of cost normalized by the q^{th} quantile converges to a Weibull with parameters θ and $\lambda = e^{1-q}$, i.e.,

$$\lim_{t\to\infty} F(z\ q(t), t) = \exp\left(-\lambda z^{1/\theta}\right). \tag{38}$$

and the q^{th} quantile of the cost distribution decreases at an asymptotically constant rate α/θ, i.e.,

$$\lim_{t\to\infty} \frac{1}{q(t)} \frac{\partial q(t)}{\partial t} = -\frac{\alpha}{\theta}. \tag{39}$$

The proof of this Proposition uses the following Lemma.

Lemma A.5: If the cdf $1-G(z)$ has an elasticity that is continuous, strictly positive, and finite at zero equals to $\epsilon(0)$, then

$$\lim_{z\to 0} \frac{G'(kx)}{G'(x)} = k^{\epsilon(0)-1}, \text{ for all } k > 0.$$

Proof: Using the definition of the elasticity and letting $k < 1$ (a similar argument applies for the case $k > 1$)

$$\frac{G'(kz)}{G'(z)} = \frac{1}{k} \frac{\epsilon(kz)}{\epsilon(z)} e^{-\int_{kz}^{z} \frac{\epsilon(u)}{u} du}.$$

From the continuity of the elasticity we know that for every ς there exist a z such that $\epsilon(0) - \varsigma \leq \epsilon(u) \leq \epsilon(0) + \varsigma$ for all $u < z$. Therefore,

$$k^{\epsilon(0)+\varsigma-1} \leq \frac{G'(kz)}{G'(z)} \leq k^{\epsilon(0)-\varsigma-1}.$$

Since ς can be made arbitrarily small, we obtain the desired result. □

Proof of Proposition A.4. Taking the limit as $t \to \infty$ in both sides of equation (38)

$$\begin{aligned}\lim_{t\to\infty} F(z\ q(t),t) &= \lim_{t\to\infty} \exp\left[\log(G(z\ q(t)))e^{\alpha t}\right] \\ &= \exp\left[\lim_{t\to\infty} \frac{G'(z\ q(t))z\ q'(t)}{-\alpha e^{-\alpha t}}\right] \\ &= \exp\left[\lim_{t\to\infty} -e^{1-q}\frac{G'(z\ q(t))}{G'(q(t))}z\right] \\ &= \exp\left(-\lambda z^{1/\theta}\right),\end{aligned}$$

where the third equality uses that $q'(t) = \alpha \log\left[G(q(t)\right] G(q(t))/G'(q(t))$, which itself follows from applying the implicit function theorem to equation (37), and the last equality follows from the following Lemma. Finally, we derive equation (39)

$$\begin{aligned}\lim_{t\to\infty} \frac{\frac{\partial q(t)}{\partial t}}{q(t)} &= \lim_{t\to\infty} \frac{\frac{\alpha \log(G(q(t))G(q(t))}{G'(q(t))}}{q(t)} \\ &= -\alpha \lim_{t\to\infty} \frac{log(G(q(t))}{1-G(q(t))}\left(\frac{1-G(q(t))}{-G'(q(t))q(t)}\right) \\ &= -\frac{\alpha}{\theta},\end{aligned}$$

where the first equality follows from applying the Implicit Function Theorem to equation (37), and the last equality following from L'Hopital rule to the first term and the fact that the elasticity at zero of the c.d.f. equals $1/\theta$. □

This result describes the asymptotic behavior of economies whose initial distribution of cost has a strictly positive and finite elasticity at zero but do not satisfy the condition in Proposition A.3, e.g., the distribution in last example described in Section 2. Proposition A.3 shows that in these economies costs decrease asymptotically at a constant rate α/θ. Nevertheless, in these economies the distribution of costs normalized by a constant growth factor, $e^{\frac{\alpha}{\theta}t}$, is asymptotically degenerate. What happens in this example is that along most of the transition costs decrease a rate that is bounded away from α/θ.

Notice that Proposition A.3 is very related to results in the mathematical statistic literature on extreme distributions. In particular for the maximum of an infinite sequence of iid variables with finite upper bound. In that case the conditions on the elasticity is essentially the same as the von Mises condition, and the invariant distribution is Weibull. See, for example, Theorem 3.3.12 of Embrechts et al. (2003). Yet our result is different in an important way from the standard results in extreme distributions. In our set-up we obtain *geometric* growth, while, in the language of the extreme distributions, the standard result is a *linear* norming constraint, or linear growth in term of economics. Indeed the standard set-up in the mathematical statistical literature is closer to the set-up in economic models of diffusion of technologies with an exogenous idea source, such as the one by Kortum (1997). In these type of model there is no growth asymptotically. Furthermore, since in our framework there is growth asymptotically we need to focus in a stronger notion of convergence. This leads to a smaller set of initial distribution that are stable, i.e., those satisfying condition (11).

A.5 Interpretation of the Continuous Time Limit

For some readers the continuous time law of motion of $F(z,t)$ may seem odd, since for small Δ there are a "fractional" number of meetings. Here we show that our limit as Δ goes to zero can be regarded as simply an "extrapolation" of the law of motion to all values of t, with no change on the substance -provided the value of α is adjusted accordingly-, but with a simpler mathematical formalization. To see this consider the following discrete time law of motion for the right CDF of a close economy:

$$F(z,j+1) = F(z,j)F(z,j) = F(z,j)^2, \text{ for all } j = 0,1,2,3,...$$

where we are measuring time in units so that there is *exactly* one meeting per period. In this case j is also the number of meetings since time zero. Continuing this way, and taking logs

$$\log F(z,j+1) = 2\log F(z,j) = 2^j \log F(z,0)$$

If we now measure time t in natural units (say years) and we assume that there are α' meetings per unit of time, we can write that j periods correspond to $t = j/\alpha'$ (years) and replacing in the previous expression

$$\log F(z,j) = 2^{\alpha' t} \log F(z,0)$$

Compare this with the continuous time limit we obtain in Section 2:[14]

$$\log F(z,t) = e^{\alpha t} \log F(z,0)$$

[14]To be more precise, $\log \tilde{F}(z,t) = \log F(z, t/\alpha'^{\alpha t}) = e^{\alpha t} \log F(z,0) = e^{\alpha t} \log \tilde{F}(z,0)$.

Thus both expression for the law of motion give identical expression (on integers values of t/α) if

$$\log(2) = \alpha/\alpha'.$$

In other words, the continuous time value of α has to be smaller than the discrete time value to take into account the "compounding" effect of the meetings, but otherwise they give the same answer.

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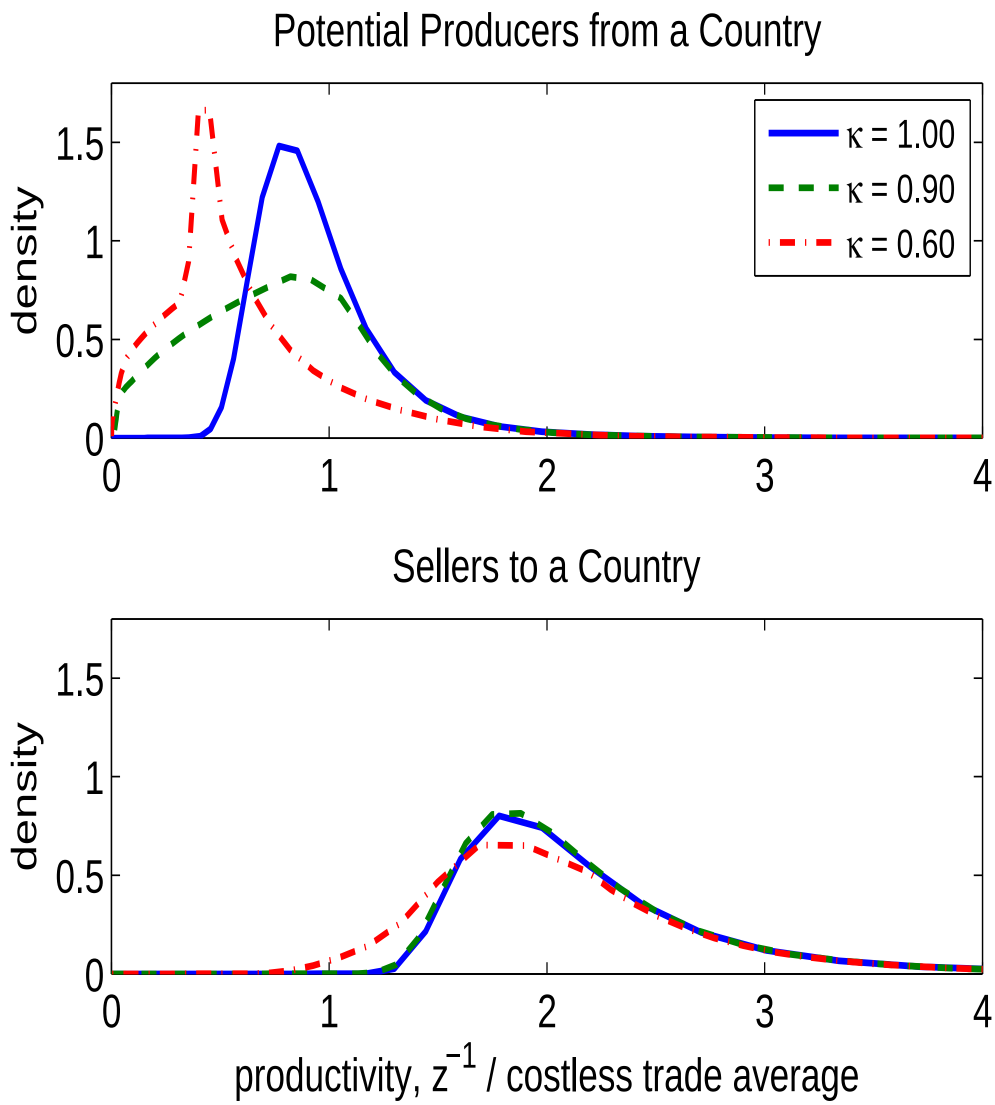
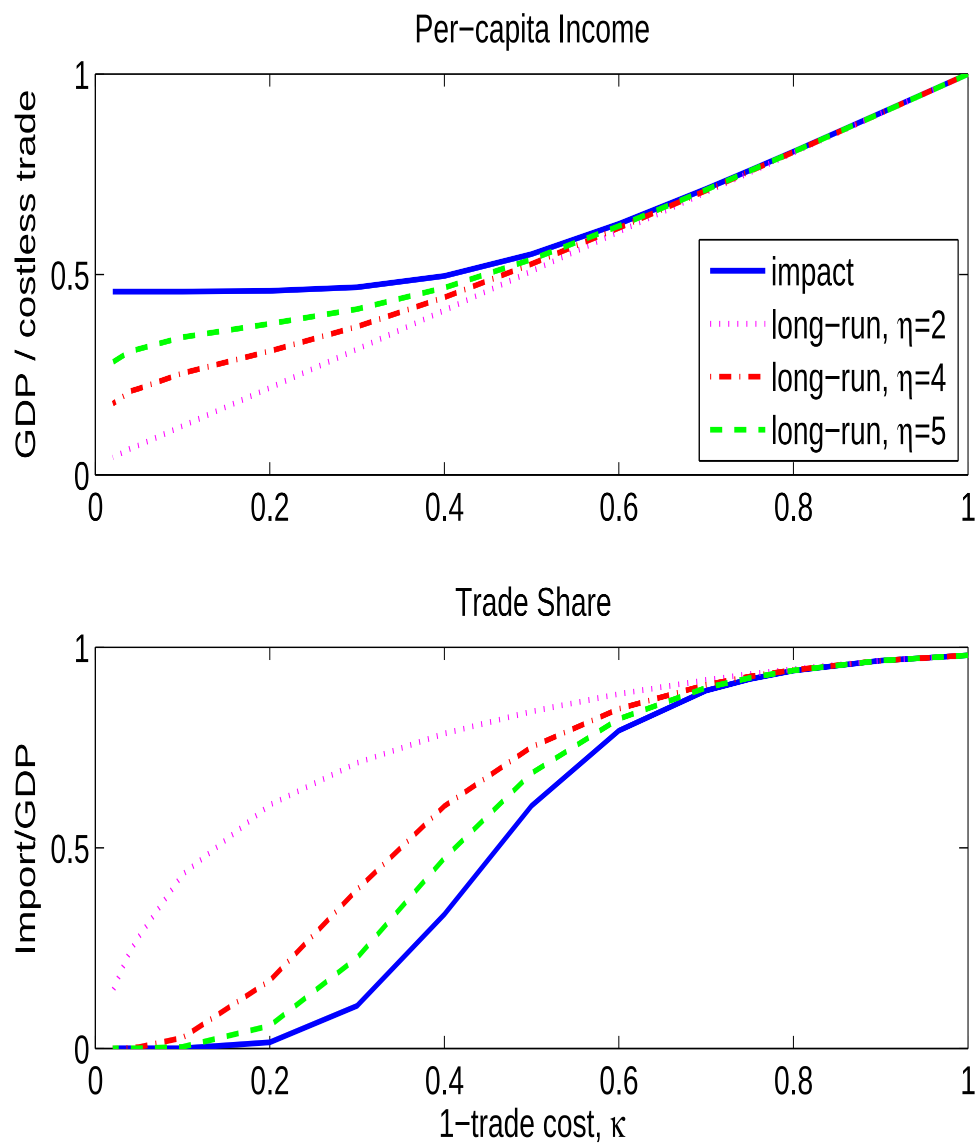
Figure 1: Long Run Effect of Trade Cost on the Distribution of Productivity



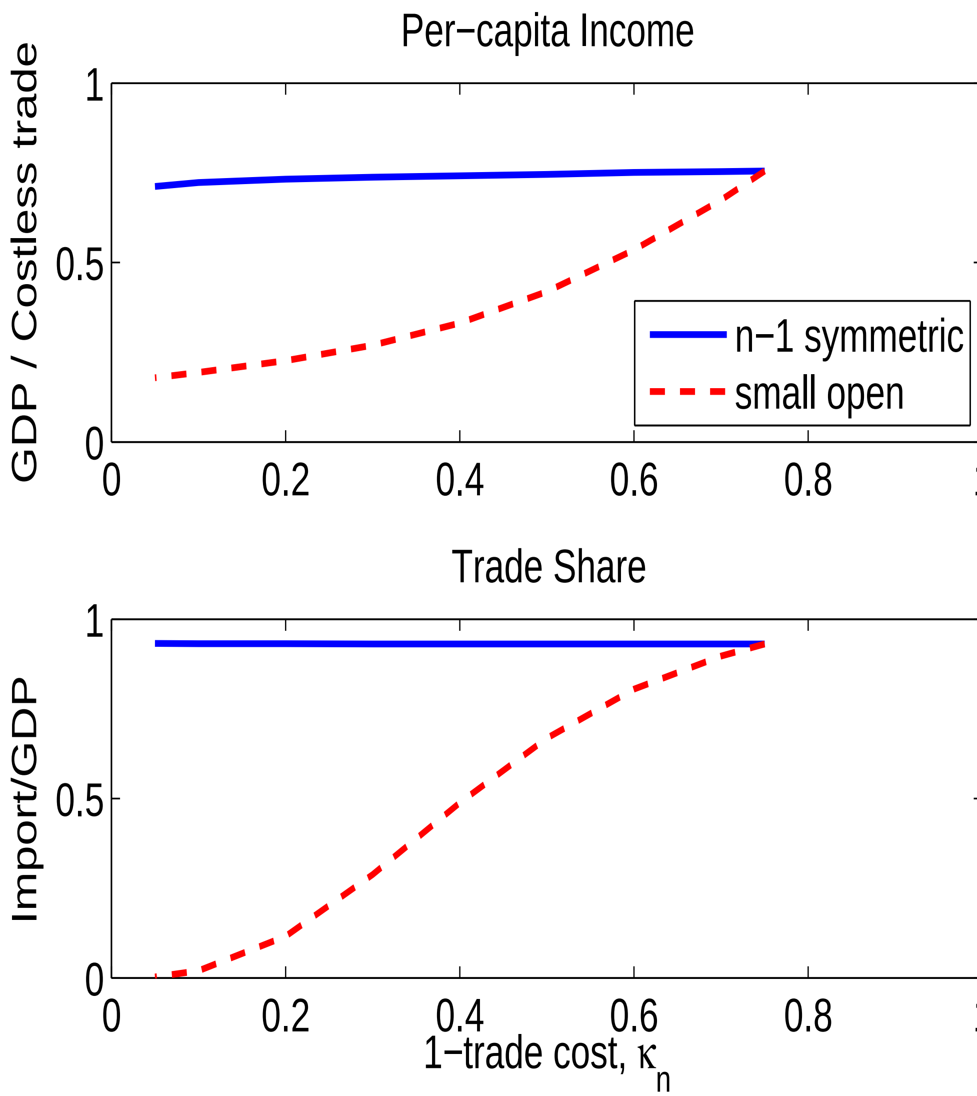
The top panel displays the density of the stationary distribution of normalized productivity $1/z$ of potential producers in a country. for different value of trade cost, $\kappa = 1, 0.9, 0.6$. The bottom panel displays the stationary density of normalized productivities for the in each country. The productivities in the x-axis are measured relative to the expected value of the stationary distribution of potential producers in the case of costless trade. We consider a world economy with $n = 50$ symmetric locations with parameter values $\theta = 0.20$, $\alpha = 0.002$.

Figure 2: Impact and Long run effect of introducing trade cost



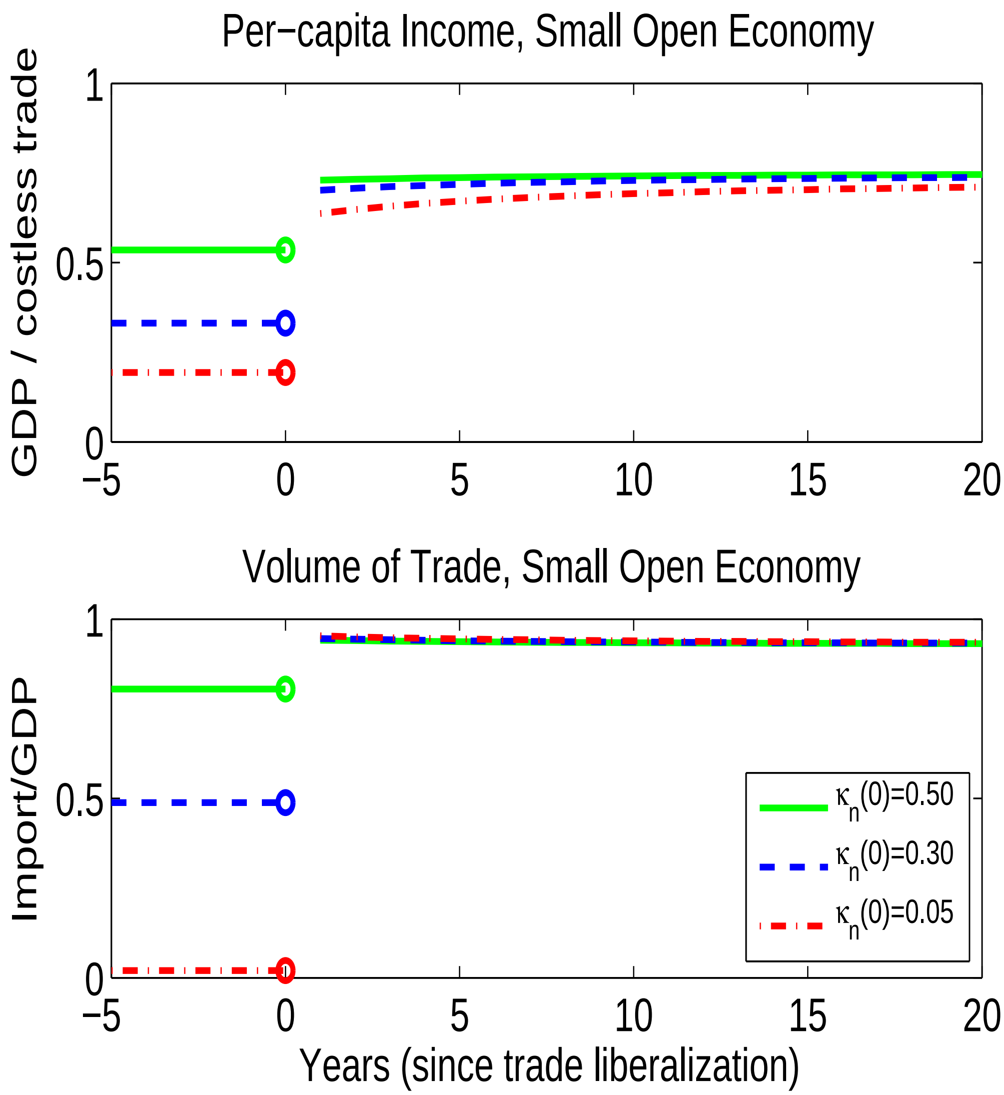
The top panel shows the effect of trade cost on per-capita income on impact (solid), and the long-run effect for various values of $\eta = 2, 4, 5$. The comparison, and the initial condition, is given by the model with costless trade, i.e., $\kappa = 1$. The bottom panel shows the effect of trade cost on the volume of trade, measured as import to GDP. The effect on impact is the same regardless of the value of η. We consider a world economy with $n = 50$ symmetric locations with parameter values $\theta = 0.20$, and $\alpha = 0.002$.

Figure 3: Long run effect of increasing trade costs in a small open economy



The top panel shows the effect of increasing the trade cost of a small open economy, κ_n, on per-capita income. The trade cost of the $n-1$ remaining countries is fixed at $\kappa_1 = 0.75$. The solid line shows the effect on the remaining $n-1$ symmetric countries. The dashed line shows the effect on the n^{th} small open economy. The per capita income are compared with the value they would have have if there trade cost will be zero in all countries, i.e. $\kappa_1 = \kappa_n = 1$. The bottom panel shows the effect of increasing the trade cost of the small open economy on the volume of trade, measured as imports to GDP. We consider a world economy with $n = 50$ countries. We use $\theta = 0.20$, $\alpha = 0.002$, $\eta = 3$.

Figure 4: Transitional Dynamics Following a Reduction in Trade Cost of a Small Open Economy



The top panel shows the dynamics of per-capita GDP of the n^{th} originally closed country following a reduction in trade cost, $\kappa_n(0) \to \kappa_n = 0.75$, for three initial levels of trade cost, $\kappa_n(0) = 0.05, 0.30, 0.50$. The trade cost of the $n-1$ symmetric countries is fixed at $\kappa_1 = 0.75$. Per-capita GDP is measured relative to the value in a world with costless trade, i.e., $\kappa_1 = \kappa_n = 1$. The bottom panel shows the corresponding dynamics of the volume of trade, measured as imports to GDP. We consider a world economy with $n = 50$ countries. We use $\theta = 0.20$, $\alpha = 0.002$, $\eta = 3$.