Optimal inflation in a model of inside money

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Abstract

There are several models of outside money in which some inflation accomplished through lump-sum transfers is optimal. It is shown here that inflation can be optimal in a model of inside money, essentially the model in Cavalcanti-Wallace (1999). The possibility of inflation comes about via the trades between people who can issue inside money, monitored people, and those who cannot, nonmonitored people. Inflation occurs at the optimum if the monitored people spend more in such meetings when they are buyers than they receive in such meetings when they are sellers.

Key words: inside-money, inflation, monitoring, optima

JEL classification: E52, E58

1 Introduction

There are, by now, several models of outside money in which some inflation produced by lump-sum transfers is optimal (see Levine [7], Kehoe et al [5], Molico [8], Green and Zhou [4], Deviatov [3].) In those models, the transfers have a beneficial effect on extensive margins by altering the money holdings of those who trade in a way that more than offsets their harmful effect on intensive margins implied by the decrease in the return on money. We study optima in a model of inside money, essentially the model in Cavalcanti and Wallace [2], and show, by way of examples, that inflation can also be optimal in that model.

The possibility of inflation in our model comes about via the trades that occur in meetings between people who can issue inside money, monitored people, and those who cannot, nonmonitored people. Inflation occurs if the monitored people spend more in such meetings when they are buyers than they receive in such meetings when they are sellers. The role of inflation bears some resemblance to its role in the above models of outside money. Although individual

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money holdings are restricted to be in the set $\{0, 1\}$, inflation allows the post-trade distribution of money holdings to differ from the pre-trade distribution in a way that turns out to be valuable in terms of ex ante welfare—representative-agent welfare before both monitored status and initial money holdings are assigned.

Before turning to the model, we should say a word about how we interpret *inside money* and about the meaning of *inflation* in a model in which individual money holdings are in the set $\{0, 1\}$. Inside money can be interpreted as trade-credit instruments (or so-called company scrip) that is (i) issued by monitored people when they buy from nonmonitored people; (ii) used by nonmonitored people in trade among themselves; and (iii) redeemed by *all* monitored people when they sell to nonmonitored people—as in a network of banks, each of which accepts the banknotes issued by other banks. As regards inflation, in models with divisible money, a standard normalization holds the stock of money fixed and represents inflation by a proportional tax on money holdings. Our approach is the same, except that the discreteness of money forces us to use a probabilistic version of such a tax: a person who ends up after trade with a unit of money loses it with some probability.

2 The model

Time is discrete and there is a nonatomic measure of people each of whom maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. Production and consumption occur in pairwise meetings that occur at random in the following way. Just prior to such meetings, each a person looks forward to being a consumer (a buyer), who meets a random producer (seller) with probability $\frac{1}{K}$, looks forward to being a producer who meets a random consumer with probability $\frac{1}{K}$, and looks forward to no pairwise meeting with probability $1 - \frac{2}{K}$, where $K \geq 2$. The period utility of someone who becomes a consumer and consumes $y \in \mathbb{R}_+$ is $u(y)$, where u is strictly increasing, strictly concave, differentiable, and satisfies $u(0) = 0$. The period utility of someone who becomes a producer and produces $y \in \mathbb{R}_+$ is $-c(y)$, where c is strictly increasing, convex, and differentiable and $c(0) = 0$. In addition, $y^* = \arg\max_{y \geq 0}[u(y) - c(y)]$ is positive. Production is perishable; it is either consumed or lost.[1]

People in the model are ex ante identical but the fraction α become permanently monitored (m people), while the rest are permanently nonmonitored (n

[1] This formulation is borrowed from Trejos and Wright [11] and Shi [9]. If K is an integer that exceeds two, then, as is well-known, it can be interpreted as the number of goods and specialization types in those models.

people). For m people, histories and money holdings are common knowledge; for n people, they are private. However, the monitored status and consumer-producer status of people in a pairwise meeting are common knowledge. And, no one except the planner can commit to future actions.[2]

Each person and the planner have printing presses capable of turning out identical, indivisible, and durable objects. Those turned out by the printing press of any one person are, however, distinguishable from those turned out by other peoples' printing presses. That allows us to prevent an m person who defects from issuing additional money. Finally, each person's holding of money issued by others is restricted to be in $\{0, 1\}$.

3 Implementable allocations and the optimum problem

We limit the search for an optimum to allocations that are steady states and symmetric. By symmetry, we mean that all people in the same situation take the same action, an action that could be a lottery, and that all monies issued by m people who have not defected and any money issued by the planner are treated as perfect substitutes.[3] We assume that all monies issued by n people are worthless. The detailed specification of the planner's problem is set out in the Appendix.

The planner's objective is ex ante expected utility, where α is the probability of becoming an m person and where the probabilities of starting with money are given by the fractions of m and n people with a unit of money—fractions which the planner chooses. The planner also chooses trades in meetings (as a function of the states of the producer and the consumer in the meeting); and chooses the inflation rate (the probability that money held after meetings is lost). We also permit the planner to make transfers of money. The planner is constrained by the steady-state restriction and by self-selection constraints that follow from our specification of private information and of punishments.

We assume that the only feasible punishment is permanent banishment of an individual m-person to the set of n-people, which includes loss of the ability to issue money. Underlying this assumption is free exit at any time from the set

[2] This part of the model is borrowed from Cavalcanti and Wallace [2].

[3] Given a bound on discrete money holdings, trade can be enhanced by distinguishing among monies—say, by color (see Aiyagari et al [1]). Here we ignore that possibility because the only role of restricting money holdings to $\{0, 1\}$ is to limit the number of unknowns in our optimum problem. If we permit distinctions among monies, then that role is compromised.

Also, although the planner could, in principle, randomize and treat some m people as n people, that is never optimal (see Wallace [12]).

of m-people into the set of n people and the ruling out of global punishments—like the shutting down of all trade in response to individual defections. We also allow both individual defection (IR) and cooperative defection by the pair in any meeting.

There are several general features of an optimum that hold in all our examples. First, m people never hold money issued by others before meetings—a general result (see Wallace [12]). This means that when an m person collects money in a meeting, that m person destroys it or turns it in the planner. (We, of course, check that the m person, who cannot hide money, is willing to do that.) Also, it is not optimal for the planner to make transfers of money, transfers which could go only to n people who end meetings without money. Finally, aside from the steady-state condition, the only constraints that turn out to be binding in our examples are IR seller constraints.

4 Examples

In order to learn a bit about the properties of optima, we compute optima for some examples.[4] Our examples are arbitrary except in two respects. First, if everyone were monitored ($\alpha = 1$), then the first-best would be implementable. (That is, only the presence of n people prevents the first-best outcome from being attained.) Second, if no one were monitored ($\alpha = 0$), then it would be desirable to pay interest on money if that were feasible. We will express these conditions as restrictions on the discount factor, β, for given u, c, and K.

In this economy, the best ex ante outcome subject only to physical feasibility is production and consumption equal to y^* in each meeting, the output that maximizes *surplus* in a meeting. If $\alpha = 1$, then y^* in every meeting is implementable if and only if

$$\frac{u(y^*)}{c(y^*)} \geq 1 + K(1-\beta)/\beta. \tag{1}$$

(This assures that a producer in a meeting weakly prefers producing y^* to permanent autarky given that others will produce y^* in future meetings. Notice

[4] The optimum problem is solved using the General Algebraic Modeling System (GAMS), which is designed for the solution of large linear, nonlinear, and mixed integer optimization problems. It consists of a language compiler and a large menu of stable integrated high-performance solvers. The solvers are divided into two groups: local solvers (which are fast, but do not guarantee that the global solution is located) and global solvers (which are slow, but are very likely to find the global optimum). The global solver used is a Branch-And-Reduce Optimization Navigator (BARON) solver. BARON uses a deterministic algorithm of the branch-and-bound type, which is guaranteed to find the global optimum under very general conditions. These conditions include bounds on variables and the functions of them that appear in the nonlinear programming problem to be solved.

that the defection option is permanent autarky when $\alpha = 1$.) If β^* denotes the β for which (1) holds at equality, then we want $\beta \geq \beta^*$.

If, instead, $\alpha = 0$, then trade occurs only when the producer has no money and the consumer has money, a *trade* meeting. The output level y^* in every trade meeting is implementable if and only if

$$\frac{u(y^*)}{c(y^*)} \geq 1 + \frac{K(1-\beta)/\beta}{1-\theta}, \tag{2}$$

where θ is the fraction with a unit of money. (This says that a producer in a meeting weakly prefers producing y^* and acquiring money with probability 1 to autarky given that others will do so in future meetings.) Let β^{**} be the value of β for which (2) holds at equality when $\theta = 1/2$, the magnitude of θ that maximizes the frequency of trade meetings. (Obviously, $\beta^{**} > \beta^*$.) As is well-known, if $\beta \geq \beta^{**}$, then the optimum is $y = y^*$ and $\theta = 1/2$. If, however, $\beta < \beta^{**}$, then the optimum has $\theta < 1/2$, $y < y^*$, and (2) at equality. In that case, it would be desirable to pay interest on money if doing so were feasible.[5] In accord with both restrictions, we set $\beta = (\beta^{**} + \beta^*)/2$.

Throughout, we fix u, c, and K as follows: $u(y) = 1 - e^{-10y}$, $c(y) = y$, and $K = 3$.[6] (This specification implies that $y^* = (\ln 10)/10 \approx .23$ and that $\beta = .59$.) We report results for $\alpha \in \{1/4,\ 1/2,\ 3/4\}$.

We start with a description of welfare and of some aggregates (see Table 1). Welfare is reported relative to ex ante welfare in the first-best allocation; namely, $[u(y^*) - c(y^*)]/[K(1-\beta)]$.

Table 1. Aggregates

	$\alpha = 1/4$	$\alpha = 1/2$	$\alpha = 3/4$
ex ante welfare	.233	.326	.431
pre-meeting welfare, m	.380	.432	.488
pre-meeting welfare, n without money	.100	.113	.133
pre-meeting welfare, n with money	.358	.401	.458
pre-meeting fraction of n with money	.299	.371	.398
inflation rate	.082	.104	.114

Ex ante welfare, the first row, is the outcome for the planner's objective. As expected, it is increasing in α. It turns out that m people realize higher welfare than n people. Also, the fraction of n people with money is substantially less

[5] With only n people, having $\theta < 1/2$ is the only way to loosen constraint (2). It does so by reducing the expected number of periods during which money is held before a trading opportunity occurs and comes at the expense of a reduction in the frequency of trade meetings.

[6] We chose this functional form for u mainly because the optimization program works better with a finite marginal utility at zero. Otherwise, it is completely arbitrary, as is the parameter that determines $u'(0)$, except that we want it to be sufficiently large relative to $c'(0)$.

than one-half—in part, because of our choice of the discount factor. Finally, the inflation rate is positive.

The positive inflation rate is best understood by examining features of the trades that occur. The next table shows output (relative to the first best level) in each kind of meeting in which positive output is possible.

Table 2. Output in meetings (y/y^*)			
(producer)(consumer)	$\alpha = 1/4$	$\alpha = 1/2$	$\alpha = 3/4$
$(n0)(n1)$	.606	.663	.739
$(n0)(m)$	.606	.663	.739
$(m)(n0)$	.296	.141	.107
$(m)(n1)$	.717	.818	.911
$(m)(m)$	.717	.818	.911

The table does not show the transfer of money from the consumer to the producer because that turned out to be extremely simple. Although we allow for lotteries, they are not optimal for these examples. In the first-, second-, and fourth-row meetings, a unit of money is transferred from the consumer to the producer. In the third- and fifth-row meetings, no money is transferred. In every meeting, other than that in the third row, the producers's IR constraint is binding.

When the producer is an n person (the first two rows), output (the level) is $\beta[(1-\xi)v^{n1} - v^{n0}]$, where ξ is the inflation rate and v^{ni} is pre-meeting welfare (the level) of an n person with i units of money. In other words, those trades are take-it-or-leave-it offers by the consumer. As in a version with $\alpha = 1$, when the producer is an m person (the last three rows), production is always a *gift* that is sustained by the threat of banishment to the set of n people without money. In the meetings in the last two rows, output (the level) is $\beta(v^m - v^{n0})$, where v^m is welfare (the level) of an m person. Although no constraint is binding in the third-row meeting, a larger output there would lower v^m and raise v^{n0}, and, therefore, lead to a violation of the *IR* constraints for all the other meetings.

The need for inflation is implied by the expenditures of money in the second- and fourth-row meetings. The inflow into money holdings of n people occurs via the second-row meeting, and it is proportional to the fraction of n people without money. The outflow from money holdings of n people occurs via the fourth-row meeting, and it is proportional to the fraction of n people with money. Because the fraction of n people with money is less than one-half, the inflow exceeds the outflow, which makes inflation necessary.

The inflow into money holdings of n people could be reduced by having a lottery in the second-row meeting—by having money go to the producer only with some probability. However, that would reduce output in that meeting.

And, although inflation gives rise to a lottery over whether the producer ends up with money, the two lotteries are not equivalent because the inflation lottery hits every n person with money at the end of trade.

Finally, notice that the price of goods in the first- and second-row meetings is higher than that in the fourth-row meeting. In other words, an n person faces a lower average price as a buyer than as a seller. Other things equal, this makes the acquisition of money by n people more desirable.

5 Concluding remarks

How representative are the above examples? It should be obvious from our description of the choice of parameters that the optimality of inflation is generic in the sense that it holds in an open set in the parameter space. We also suspect that it is robust in other senses.

In particular, consider a richer set of individual money holdings: $\{0, 1, 2, ..., B\}$. Absent inflation or deflation, any steady state must have a zero net flow of money arising from trades between monitored people and nonmonitored people. Given that the first best is unachievable in this model (see Wallace [12]), it is reasonable to surmise that an optimum with a zero net flow is nongeneric. The $\{0, 1\}$ ($B = 1$) case is, however, special in not permitting an outcome that resembles deflation. With $B > 1$, deflation can be approximated by having monitored people earn more than they spend in meetings with nonmonitored people and by having the excess returned to nonmonitored people roughly in proportion to their holdings—roughly because proportionality has to be approximated probabilistically and because those at the upper bound must be excluded. In the $B = 1$ case, the excess can only go to those with 0, which does not approximate deflation.

So what might we find for $B > 1$? In a model with $\alpha = 0$, $B = 2$, and the same notion of implementability used here, Deviatov [3] finds some examples in which inflation produced by an approximation to lump-sum transfers is optimal and others in which zero inflation is optimal. (He cannot produce deflation because there is no one to tax in his model.) We suspect that the positive-α model with $B = 2$ will give rise to optimal inflation in more examples for two reasons: first, spending by monitored people tends to be a good thing in the model; second, as we saw in the examples above, money holding by nonmonitored people can be rewarded through different prices in the different kinds of meetings.

Finally, as is true for existing models of outside money, it would seem that inflation cannot improve outcomes if the distribution of money holdings is degenerate (as in, for example, Lagos and Wright [6] and Shi [10]). In such models, there is only an intensive margin and that margin is necessarily worsened by inflation.

6 Appendix

We start with some notation. Let $S = \{m, n\} \times \{0, 1\}$ be the set of individual states, where $s = (s_1, s_2)$ and $s' = (s'_1, s'_2)$ denote generic elements of S. The state of a meeting is denoted (s, s'), where s is the state of producer and s' is the state of the consumer. Our main notation is summarized in Table 3.

Table 3. Notation	
$y^{s,s'}$	production by s and consumption by s' in a meeting
$\lambda_p^{s,s'}(i)$	prob. that end-of-meeting money is i for producer
$\lambda_c^{s,s'}(i)$	prob. that end-of-meeting money is i for consumer
ξ	prob. that money is lost after meetings
$\phi^s(i)$	prob. that next-date money is i when today's state is s
θ^s	fraction in state s before meetings
v^s	discounted utility for s before meetings

The planner chooses the variables in the table subject to the constraints set out below.

6.1 Feasibility and steady state conditions

The λ's and ϕ's must, of course, be lotteries on $\{0, 1\}$ and must satisfy the obvious feasibility restrictions; in particular, money is not created in a meeting between two n people.

Trades and transfers imply the following transition probabilities of a person's money holding from the start of one date to the start of the next date. The probability that a person in state $(s_1, i) \in S$ transits to state $(s_1, j) \in S$ is

$$t^{s_1}(i,j) = \frac{1}{K} \sum_{s' \in S} \theta\left(s'\right) [(\lambda_p^{(s_1,i),s'} + \lambda_c^{s',(s_1,i)} + (K-2)\delta_i)\Psi\Phi^{s_1}(j)], \tag{3}$$

where $\lambda_p^{s,s'} = (\lambda_p^{s,s'}(0), \lambda_p^{s,s'}(1))$, $\lambda_c^{s,s'} = (\lambda_c^{s,s'}(0), \lambda_c^{s,s'}(1))$, δ_i is the two-element unit vector in direction $i + 1$,

$$\Psi = \begin{bmatrix} 1 & 0 \\ \xi & 1-\xi \end{bmatrix}, \tag{4}$$

and $\Phi^{s_1}(j) = (\phi^{(s_1,0)}(j), \phi^{(s_1,1)}(j))'$. If T^{s_1} denotes the 2×2 matrix whose (i,j)-th component is $[t^{s_1}(i,j)]$, then the steady state requirements are

$$(\theta^{(s_1,0)}, \theta^{(s_1,1)})T^{s_1} = (\theta^{(s_1,0)}, \theta^{(s_1,1)}), \tag{5}$$

for $s_1 \in \{m, n\}$, where $\theta^{(m,0)} + \theta^{(m,1)} = \alpha$ and $\theta^{(n,0)} + \theta^{(n,1)} = 1 - \alpha$.

6.2 Incentive constraints

It is convenient to first define discounted expected utility before meetings. For $s \in S$, we have

$$v^s = \frac{1}{K} \sum_{s' \in S} \theta^{s'} [\pi^p(s, s') + \pi^c(s', s) + (K-2)\pi^0(s)], \tag{6}$$

where

$$\pi^p(s, s') = -c(y^{s,s'}) + \beta \lambda_p^{s,s'} \Psi \Phi^{s_1} v^{s_1}, \tag{7}$$

$$\pi^c(s, s') = u(y^{s,s'}) + \beta \lambda_c^{s,s'} \Psi \Phi^{s'_1} v^{s'_1}, \tag{8}$$

and

$$\pi^0(s) = \beta \delta_{s_2} \Psi \Phi^{s_1} v^{s_1}. \tag{9}$$

Here, δ_{s_2} is the 1×2 unit vector in direction $s_2 + 1$,

$$\Phi^{s_1} = \begin{bmatrix} \phi^{(s_1,0)}(0), \phi^{(s_1,0)}(1) \\ \phi^{(s_1,1)}(0), \phi^{(s_1,1)}(1) \end{bmatrix}, \tag{10}$$

and $v^{s_1} = (v^{(s_1,0)}, v^{(s_1,1)})'$. Given the variables in the first six rows of Table 3, Blackwell's sufficient conditions for contraction imply that v^s exists and is unique. We express the incentive constraints in terms of the v's. This is legitimate because the principle of one-shot deviations applies to this model.

There are truth-telling constraints only for n people with money when they are consumers. They are

$$\pi^c(s, (n, 1)) \geq u(y^{s,(n,0)}) + \beta(\xi, 1-\xi)\Phi^n v^n. \tag{11}$$

This potentially binds only when $s_1 = m$, when the producer is an m person.

The individual rationality constraints for meetings are

$$\pi^p((s_1, 0), s') \geq \beta \phi^{(n,0)} v^n \text{ and } \pi^p((s_1, 1), s') \geq \beta(\xi, 1-\xi)\Phi^n v^n, \tag{12}$$

$$\pi^c(s,(s_1',0)) \geq \beta\phi^{(n,0)}v^n \text{ and } \pi^c(s,(s_1',1) \geq \beta(\xi,1-\xi)\Phi^n v^n, \tag{13}$$

and

$$\pi^0((s_1,0),s') \geq \beta\phi^{(n,0)}v^n \text{ and } \pi^0((s_1,1),s') \geq \beta(\xi,1-\xi)\Phi^n v^n. \tag{14}$$

We also have a constraint which says that m people prefer the transfers intended for them to defecting to n-status just prior to those transfers; namely,

$$\phi^{(m,s_2)}v^m \geq \phi^{(n,s_2)}v^n. \tag{15}$$

There is also a constraint that transfers to n people are nonnegative.

We also allow cooperative defections for people in meetings. These constraints, which turned out not to be relevant for our examples, are that the trades be in the pairwise core for those in a meeting when they take as given the continuation values.

6.3 The planner's problem

The planner chooses the variables in Table 3 to maximize

$$\sum_{s\in S}\theta^s v^s = \frac{\sum\sum_{s\in S,s'\in S}\theta^s\theta^{s'}[u(y^{ss'})-c(y^{ss'})]}{K(1-\beta)} \tag{16}$$

subject to all the relevant constraints.

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