A Theory of Education and Health

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Abstract

This paper presents a unified theory of human capital with both health capital and, what we term, skill capital endogenously determined within the model. By considering joint investment in health capital and in skill capital, the model highlights similarities and differences in these two important components of human capital. Health is distinct from skill: health is important to longevity, provides direct utility, provides time that can be devoted to work or other uses, is valued later in life, and eventually declines, no matter how much one invests in it (a dismal fact of life). The theory provides a conceptual framework for empirical and theoretical studies aimed at understanding the complex relationship between education and health, and generates new testable predictions on (i) the effect of health on skill formation, and (ii) the powerful effect of longevity gains on health and economic inequality.

Keywords: health investment; lifecycle model; human capital; health capital; optimal control

JEL Codes: D91, I10, I12, J00, J24

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1 Introduction

The United States' 20^{th} Century was characterized by unprecedented increases in economic growth, with real per capita income in 2000 five to six times its level in 1900 (Goldin and Katz, 2009). The 20^{th} Century additionally differentiated itself by significant increases in life expectancy, health, and educational attainment. Life expectancy at birth increased by about 30 years, from 46 years in 1900 to 74 in 2000 for white men (Centers for Disease Control and Prevention; cdc.gov), and years of schooling rose from seven years in 1900 to 13 years in 2000 (Bleakley, Costa, and Lleras-Muney, 2013). Similary impressive advances in per capita income, life expectancy, and schooling took place in other developed and increasingly also in developing nations (Deaton, 2013). While increases in life expectancy, health status, and educational attainment appear to contribute to economic growth (Barro, 2001; Bloom and Canning, 2000; Bloom, Canning, and Sevilla, 2004; Goldin and Katz, 2009),[1] it is less clear to what extent, and how, the trends in life expectancy, health, and education are related.

Studying these relations is traditionally guided by human-capital theory, the foundations of which have been laid by the seminal works of Schultz (1961), Becker (1964), Ben-Porath (1967), and Mincer (1974). What the canonical human capital model does not deny, though largely leaves out, is that human capital is multidimensional (Acemoglu and Autor, 2012). Education and health are considered to be the most critical components of human capital (Schultz, 1961; Grossman, 2000), and while they share the defining characteristic of human capital that investing in them makes individuals more productive, there are several important differences between them. Perhaps most importantly, Becker (1964) observes that investments in human capital should decrease with age as the remaining period over which benefits can be accrued decreases. While this is clearly the case for education and training decisions, investments in health generally increase with age, even after retirement when health has lost its importance in generating earnings.[2]

This and other distinctions between health and other types of human capital, identified by, e.g., Mushkin (1962), have led to the development of the so-called health-capital model by Grossman (1972a,b). While the health-capital model has been very influential in health economics, and recognizes the role of education as a productivity-enhancing factor in health investment, it treats both education and longevity as exogenous. In doing so, it leaves out the possibility that individuals jointly optimize health, longevity, and education.[3]

[1] But see Acemoglu and Johnson (2007; 2014) who suggest that gains in life expectancy generate limited or no economic growth.

[2] Investments in health consist of, e.g., medical care, physical exercise, a healthy diet and a healthy lifestyle. Not all such components of health investment necessarily increase with age. For example, the lifecycle profile of exercise is relatively flat (Podor and Halliday, 2012). But medical expenditures (e.g., Zweifel, Felder, and Meiers, 1999) and intake of fruit and vegetables do increase with age (Serdula et al. 2004; Pearson et al. 2005), and smoking rates drop with age (DHHS, 2014).

[3] Ehrlich and Chuma (1990) have included endogenous longevity in the Grossman model, and Galama

Human-capital theory (Becker, 1964; Ben-Porath, 1967) predicts a link between increases in life expectancy and educational attainment – known as the 'Ben Porath' mechanism: longevity (exogenous in the human-capital literature) increases the return to investment in human capital by lengthening the horizon over which benefits can be accrued. Health-capital theory (Grossman, 1972a;b) also predicts a link, albeit in the other direction: a causal effect of education (exogenous in the health-capital literature) on health and longevity. The argument is that the higher educated are more efficient producers of health investment through (i) more efficient use of existing inputs (productive efficiency), e.g., better management of their diseases (Goldman and Smith, 2002), (ii) use of a better mix of health investment inputs (allocative efficiency), and (iii) early adoption of new knowledge and new technology (Lleras-Muney and Lichtenberg, 2005; Glied and Lleras-Muney, 2008). Empirically, there is supporting evidence for both directions of causality.[4] Since each theory emphasizes one particular direction of causality, human-capital and health-capital theory on their own provide only partial, and often competing, explanations for the relation between education, health, and longevity.

As a result, both human-capital theory and health-capital theory fall short of providing a comprehensive framework to study the interactions between education, health, and longevity. As Michael Grossman (2000) put it *" ... Currently, we still lack comprehensive theoretical models in which the stocks of health and knowledge are determined simultaneously ... The rich empirical literature treating interactions between schooling and health underscores the potential payoffs to this undertaking ... "*.

This paper presents an explicit theory of joint investment in skill capital, health capital, and longevity, with three distinct (and endogenous) phases of life: schooling, work, and retirement.[5] Investments in health capital consist of, e.g., medical expenditures and physical exercise, while investments in skill capital consist of, e.g., expenditures on education and (on-the-job) training. Education (or schooling) is a distinct phase of life characterized by large investments in skill and limited or no work, and retirement is a distinct phase of life devoted to leisure and health investment. Individuals make their own decisions and are free to work, i.e., the start of the model corresponds to the mandatory schooling age (around 16 to 18 years for most developed nations) and the decision under consideration is whether to participate in post-mandatory education (or not) and for what

and Van Kippersluis (2015) have extended the model further by including health behaviors and the decision to accept unhealthy working conditions, which are important causes of ill-health and early mortality. Still, these models treat education as being determined outside of the model.

[4]Several studies have established a causal effect of education on health outcomes (Lleras-Muney, 2005; Conti, Heckman and Urzua, 2010; Van Kippersluis, O'Donnell, and van Doorslaer, 2011), although a number of recent studies find a very small or no effect (Mazumder, 2008; Albouy and Lequien, 2009; Meghir, Palme, and Simeonova, 2012; Clark and Royer, 2014). The Ben-Porath mechanism is also supported by several studies finding a positive effect of life expectancy on skill investment (Soares, 2006; Jayachandran and Lleras-Muney, 2009; Fortson, 2011; Oster, Shoulson, and Dorsey, 2013).

[5]In order to distinguish health clearly from the traditional notion of human capital, we employ in the remainder of the paper the term "skill capital" to refer to traditional human capital, "health capital" to refer to health, and "skill-capital literature" to refer to the traditional human-capital literature.

duration.

The theory integrates (unifies) the human-capital and health-capital theoretical literatures. We are the first to develop such a comprehensive theory of education and health, and the first to investigate such a theory analytically.[6] Our analytical approach allows generating predictions that transparently follow from economic principles and assumptions, and thereby may provide guidance to structural- and reduced-form empirical analyses of skill and health production.

The theory makes two main contributions to the literature. The first contribution is of a fundamental nature: by explicitly modeling joint investment in both skill and in health, the model defines and highlights the similarities and differences in the nature of skill and health. Like skill, health is an investment good that increases individuals' productivity (Grossman, 1972a). Yet, skill and health are different and not interchangeable. In contrast to skill, health provides direct utility (Grossman, 1972a; Murphy and Topel, 2006), and health extends life (Ehrlich and Chuma, 1990). In this paper, we argue for three additional distinctions. First, skill capital (largely) determines the wage rate, while health capital (largely) determines the time spent working, both within a day by decreasing sick time (as in Grossman, 1972a), but also over the life cycle by affecting retirement and life expectancy (two essential horizons that determine the period over which investments can be recouped). Second, individuals generally start life with a healthy body, but the terminal health state is universally low (for natural causes of death it is the physically frail that eventually face the great reaper). By contrast, individuals generally start life with limited skills, but end life with various degrees of cognitive and mental fitness (some of us have the good fortune to stay mentally sharp till death). In short, skill grows while health declines. Third, skill is valued mostly early in life while health is valued mostly later in life. Thus, investments in skill are high when young, while investments in health are high when old. Hence, despite broadly similar formulations of skill- and health-capital theory, differences in initial conditions, end conditions, and production processes, lead skill and health to exhibit fundamentally different dynamics.

The second contribution of the paper consists of providing a conceptual framework to guide empirical research in human capital. The unified theory provides new insights, makes new predictions, and explains stylized facts that the individual theories of skill capital and health capital on their own cannot. We highlight the three most novel ones and discuss these more extensively in section 3.

First, while the causal effect of education on health outcomes has received much theoretical and empirical attention, the reverse causal effect from health to education has only been studied empirically: it is absent from skill-capital as well as health-capital theory.[7] The importance of this channel is illustrated by empirical studies that report a

[6]Becker (2007) develops a simple two period model of joint decisions regarding health and education. Hai and Heckman (2014) structurally estimate a dynamic lifecycle model to quantify causal effects of education on health and unhealthy behavior. For calibrated simulations of simpler multi-period models see Strulik (2013) and Carbone and Kverndokk (2014).

[7]See Bleakley (2010a) for an informal discussion of the effect of health on years of schooling. Childhood

negative effect of childhood ill-health on educational attainment (Perri, 1984; Behrman and Rosenzweig, 2004; Case, Fertig, and Paxson, 2005; Currie, 2009; Bloom and Fink, 2013). Our model not only accounts for an effect of health on educational attainment, but additionally predicts that health raises skill formation beyond the schooling period. The theoretical mechanisms that give rise to this effect are (i) health and skill are strongly complementary in generating earnings, so that an increase in health substantially raises the return to investment in skill, (ii) healthy individuals are more efficient producers of skill, and (iii) healthy individuals live longer, increasing the return to skill investment by increasing the period over which its benefits can be reaped. These three pathways from health to skill formation deserve empirical study.

Second, our model predicts a central role for longevity in explaining observed associations between wealth, skill, and health. Length of life is fixed in the traditional human- and health-capital literatures. Without ability to extend life, associations between wealth, skill, and health are absent or small. If, however, life can be extended, as in our theory, wealth, skill, and health, are positively associated and the greater the degree of life extension, the greater is their association. Thus, the ability to postpone death generates health and economic inequality. Although this provides no conclusive evidence, Figure 1 shows that, consistent with this theoretical prediction, countries with large inequality in life expectancy (a crude measure of the extent to which resources enable life extension) are also those with large inequality in education.[8]

The intuition behind this result is that the horizon (longevity) is a crucial determinant of the return to investment in skill and in health. In situations where it is difficult to increase life expectancy, associations between wealth, skill, and health, are weak because of limited returns. This may be the case for a developing nation (where there may be lack of access to basic medical care), for a nation with a high disease burden (where gains from tackling a certain disease may be limited due to the existence of other major diseases in the environment), for the developed world if it were faced with diminishing ability of technology to further extend life, or for individuals faced with Huntington's (Oster, Shoulson, and Dorsey, 2013) or other diseases that severely impact longevity.

Third, and related, our model highlights that complementarity effects, operating through longevity, reinforce the associations between wealth, skill, health, and technology. That is, the combined effect is greater than the sum of the individual effects. As an example, improvements in the productivity of skill investment reinforce the effect of life expectancy on skill formation. If skill-capital investment is relatively unproductive (e.g., low quality teachers, children infected with worms, or malaria), if the cost of skill-capital

health may impact educational attainment through (i) the physical ability to attend school, (ii) associated improved cognitive ability and thereby learning (Grantham-McGregor et al. 2007, Bleakley, 2007; Madsen, 2012), and (iii) incentivizing parents to invest more in their children's education (Soares, 2005).

[8] The Ben-Porath (1967) model cannot explain this association since length of life is assumed to be fixed and thus by definition the same for every individual . Wealthier individuals do not invest more in skill in the Ben-Porath model (Heckman, 1976) and as a result there is no inequality in education. The strength of our theory is that both life expectancy and education are endogenously determined, resulting in inequalities in life expectancy and in education.

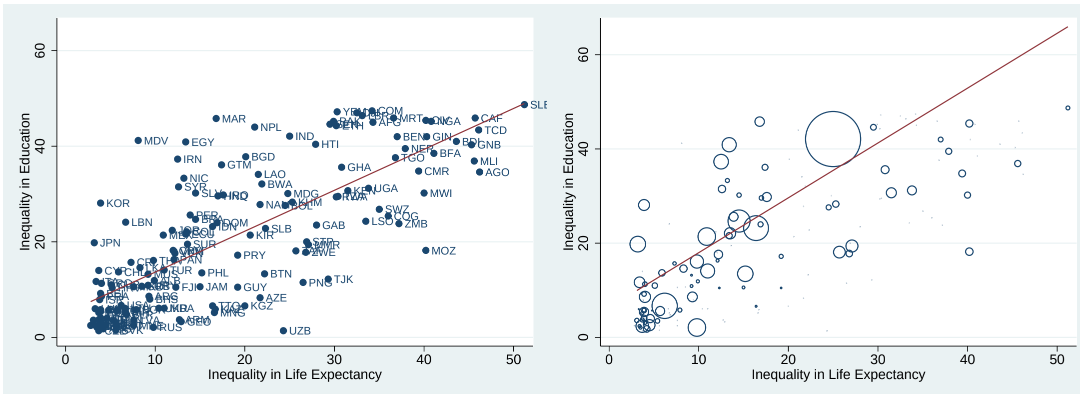

Figure 1: *Atkinson index for inequality in education versus inequality in life expectancy, unweighted (figure on the left) and weighted by population size (figure on the right). Country codes are those used by the World Bank. Source: United Nations, See Appendix A.1 for more information.*

investment is high (e.g., high tuition, long distance to schools, crops that need to be collected), or if the institutional environment generates only limited demand for skill (e.g., poor infrastructure, corruption, limited technological capabilities, etc.), then the effect of life expectancy on skill-capital formation is predicted to be modest. By contrast, when skill investment is productive, affordable, and the institutional environment is favorable, the effect of life expectancy on skill capital is predicted to be strong. This suggests there could be important heterogeneity in the effect of longevity gains on skill-capital formation, between, e.g., developed and underdeveloped nations. Indeed, Figure 2 suggests that longevity gains are associated with education gains, but strikingly not so for the least developed countries.

Complementarity also suggests that improvements in the efficiency of skill production and health production could produce multiplier effects. The United States' 20^{th} Century saw significant improvements in the productivity of health investments (e.g., clean water technologies, introduction of antibiotics) and reductions in the price of skill investment (e.g., compulsory schooling laws). It has been established that these technological and policy developments led to strong increases in life expectancy (Cutler and Miller, 2005; Lleras-Muney, 2005; Cutler, Deaton, and Lleras-Muney, 2006). Our theory suggests that the combination of (i) a higher productivity of skill and of health investment, and (ii) the associated increase in life expectancy, may have reinforced each other. Higher productivity of skill led to an increase in life expectancy, and this had a stronger effect on skill investment due to the higher productivity of skill. Jointly such complementaries may have led to high returns to investment in both skill capital and health capital, potentially explaining the unprecedented increases in skill and health during the 20^{th} Century.

These are just three examples of how the theory can be used as an analytical framework to study empirical questions and to generate testable predictions. The detailed examples

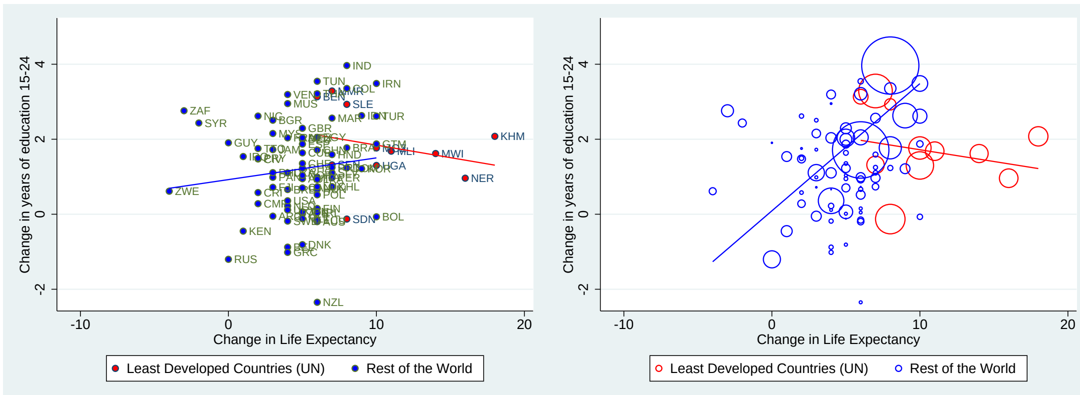

Figure 2: *Gains in years of education for the population aged 15-24 (1990-2010) versus gains in life expectancy at birth (1990-2012), unweighted (figure on the left) and weighted by population size (figure on the right). Country codes are those used by the World Bank. Source: WHO (Life Expectancy) and Barro Lee database (Education). See Appendix A.1 for more information.*

we provide in the paper, and the comparative dynamic analyses we employ to arrive at predictions, provide a template that can be followed by researchers to study their own particular research question of interest.

The paper is organized as follows. In section 2 we outline our model formulation, contrast it with the existing literature, and present first-order and transversality conditions. In section 3 we discuss the lifecycle trajectories, analyze heterogeneity in these trajectories by employing comparative dynamic analyses, and develop predictions. We conclude in section 4.

2 Model formulation and solutions

2.1 Model

The theory merges the human-capital literature with the health-capital literature. We treat health as a form of human capital that is distinct from the component of human capital that individuals invest in through education and training. We loosely refer to the latter as "skill capital" and the former as "health capital". Individuals invest in health (and longevity) through expenditures on (e.g., medical care) and time investments in (e.g., exercise) health; they invest in skill capital through outlays and time investments in skill (e.g., schooling and [on-the-job] training).

Individuals maximize the lifetime utility function

$$\mathbb{U} = \max_{X_C, L, I_E, I_H, S, R, T} \left\{ \int_0^S U[.] e^{-\beta t} dt + \int_S^R U[.] e^{-\beta t} dt + \int_R^T U[.] e^{-\beta t} dt \right\}, \qquad (1)$$

where time $t = 0$ corresponds to the mandatory schooling age (around 16 to 18 years for most developed nations), S denotes years of post-mandatory schooling (endogenous), R denotes the retirement age (endogenous), T denotes total lifetime (endogenous), β is a subjective discount factor and individuals derive utility $U[X_C(t), L(t), H(t)]$ from consumption goods and services $X_C(t)$, leisure time $L(t)$, and health $H(t)$. The utility function is increasing in each of its arguments and strictly concave.

The objective function (1) is maximized subject to the following dynamic constraints for skill capital $E(t)$ and health capital $H(t)$:

$$\frac{\partial E}{\partial t} = F_E\left[I_E(t), E(t), H(t)\right] = f_E\left[I_E(t), E(t), H(t)\right] - d_E(t)E(t), \tag{2}$$
$$\frac{\partial H}{\partial t} = F_H\left[I_H(t), E(t), H(t)\right] = f_H\left[I_H(t), E(t), H(t)\right] - d_H(t)H(t). \tag{3}$$

Skill capital $E(t)$ (equation 2) and health capital $H(t)$ (equation 3) can be improved through investments in, respectively, skill capital $I_E(t)$ and health $I_H(t)$, and deteriorate at the biological deterioration rates $d_E(t)$ and $d_H(t)$. Goods and services $X_E(t)$, $X_H(t)$, purchased in the market and own time inputs $\tau_E(t)$, $\tau_H(t)$, are used in the production of investment in skill capital and in health capital $I_E(t)$, $I_H(t)$:

$$I_E(t) = I_E[X_E(t), \tau_E(t)],$$
$$I_H(t) = I_H[X_H(t), \tau_H(t)].$$

The skill-capital F_E and health-capital F_H production processes are assumed to be increasing and strictly concave in the investment inputs $X_E(t)$, $\tau_E(t)$, and $X_H(t)$, $\tau_H(t)$, respectively.[9] Crucially, this assumption of diminishing returns to investment (concavity) addresses the degeneracy of the solution for investment that plague the health-capital literature as a result of the common assumption of constant returns to scale (see for a discussion Ehrlich and Chuma, 1990; Galama and Van Kippersluis, 2013; Galama, 2015).

The efficiencies of investment in skill capital and health capital are assumed to be functions of the stocks of skill capital $E(t)$ and of health $H(t)$. This allows us to model self-productivity, where skills produced at one stage augment skills at later stages, and dynamic complementarity, where skills produced at one stage raise the productivity of investment at later stages (Cunha and Heckman, 2007). Self-productivity can be self-reinforcing $\partial F_E/\partial E > 0$, $\partial F_H/\partial H > 0$, and/or cross fertilizing, $\partial F_E/\partial H > 0$, $\partial F_H/\partial E > 0$. Dynamic complementarity too can be self-reinforcing, $\partial^2 F_E/\partial E \partial I_E > 0$, $\partial^2 F_H/\partial H \partial I_H > 0$, and/or cross fertilizing, $\partial^2 F_E/\partial H \partial I_E > 0$, $\partial^2 F_H/\partial E \partial I_H > 0$.

The intertemporal budget constraint for assets $A(t)$ is given by

$$\frac{\partial A}{\partial t} = rA(t) + Y[t, E(t), H(t)] - p_C(t)X_C(t) - p_E(t)X_E(t) - p_H(t)X_H(t). \tag{4}$$

[9] Concavity implies $\partial^2 F_E/\partial X_E^2 < 0$, $\partial^2 F_E/\partial \tau_E^2 < 0$, $\partial^2 F_H/\partial X_H^2 < 0$, $\partial^2 F_H/\partial \tau_H^2 < 0$, $\left(\partial^2 F_E/\partial X_E^2\right)\left(\partial^2 F_E/\partial \tau_E^2\right) > \left(\partial^2 F_E/\partial X_E \partial \tau_E\right)^2$ and $\left(\partial^2 F_H/\partial X_H^2\right)\left(\partial^2 F_H/\partial \tau_H^2\right) > \left(\partial^2 F_H/\partial X_H \partial \tau_H\right)^2$.

Assets $A(t)$ (equation 4) provide a return r (the rate of return on capital) and increase with income

$$\begin{aligned} Y[t,E(t),H(t)] &= b_S(t) & 0 \le t < S, \quad (5)\\ Y[t,E(t),H(t)] &= w[t,E(t)]\tau_w[t,H(t)] & S \le t < R, \quad (6)\\ Y[t,E(t),H(t)] &= b_R(R) & R \le t < T, \quad (7)\end{aligned}$$

where during the schooling period (up to S) individuals receive a state (or parental) transfer to fund schooling $b_S(t)$ (e.g., financial aid, conditional on being in school), and during retirement individuals receive a state or private pension annuity $b_R(R)$, assumed to be a function of retirement age R. During working life (between S and R) earnings consist of the product of the wage rate $w[t,E(t)]$ and the time spent working $\tau_w[t,H(t)]$. Skill capital (largely) determines the wage rate, while health capital (largely) determines the time spent working. Assets decrease with expenditures on investment and consumption goods and services $X_E(t)$, $X_H(t)$ and $X_C(t)$, at prices $p_E(t)$, $p_H(t)$ and $p_C(t)$. Alternatively, or in addition to the schooling subsidy $b_S(t)$, the government may subsidize the cost of skill formation by reducing or fully subsidizing the price $p_E(t)$ of skill investment while in school.[10]

Finally, the total time constraint Ω is given by

$$\Omega = \tau_w(t) + L(t) + \tau_E(t) + \tau_H(t) + s[H(t)]. \quad (8)$$

During working life, the total available time Ω is divided between time spent working $\tau_w(t)$, leisure time $L(t)$, time investments in skill and in health capital $\tau_E(t)$, $\tau_H(t)$, and time lost due to illness $s[H(t)]$ (assumed to be a decreasing function of health). During school years and during retirement individuals do not work, i.e.

$$\tau_w(t) = 0. \quad (9)$$

Thus, we have the following optimal control problem: the objective function (1) is maximized with respect to the control functions $X_C(t)$, $X_E(t)$, $X_H(t)$, $L(t)$, $\tau_E(t)$, $\tau_H(t)$, and the parameters S, R, and T, subject to the constraints (2) to (8), and the following initial and end conditions: $H(0) = H_0$, $H(T) = H_T$, $E(0) = E_0$, $A(0) = A_0$, $A(T) = A_T$, and $E(T) \ge 0$ (and free). Length of life T (Grossman, 1972a;b) is determined by a minimum health level below which an individual dies: $H_T \equiv H_{\min}$.

The Lagrangian (see, e.g., Seierstad and Sydsaeter, 1987; Caputo, 2005) of this problem is:

$$\begin{aligned} \Im &= U[X_C(t),L(t),H(t)]e^{-\beta t} + q_E(t)\frac{\partial E}{\partial t} + q_H(t)\frac{\partial H}{\partial t} + q_A(t)\frac{\partial A}{\partial t} \\ &+ \lambda_{\tau_w}(t)w[t,E(t)]\tau_w(t) + \lambda_{H_{\min}}(t)\left[H(t) - H_{\min}\right], \quad (10)\end{aligned}$$

[10]We assume, for simplicity, that if an individual decides to continue her education $S > 0$, she is not allowed to work. In practice there may be attendance requirements and, depending on how stringent these are, students may have varying degrees of time available that they could devote to work for pay.

where $q_E(t)$, $q_H(t)$, and $q_A(t)$ are the co-state variables associated with, respectively, the dynamic equations (2) for skill capital $E(t)$, (3) for health $H(t)$, and (4) for assets $A(t)$, the multiplier $\lambda_{\tau_w}(t)$ is associated with the condition that individuals do not work during school years and retirement (9) ($\lambda_{\tau_w}(t) = 0$ if $\tau_w(t) > 0$ and $\lambda_{\tau_w}(t) > 0$ if $\tau_w(t) = 0$), and $\lambda_{H_{\min}}(t)$ is the multiplier associated with the condition that $H(t) > H_{\min}$ for $t < T$.

The co-state variables $q_E(t)$, $q_H(t)$, and $q_A(t)$ find a natural economic interpretation in the following standard result from Pontryagin

$$q_Z(t) = \frac{\partial}{\partial Z(t)} \int_t^{T^*} U(*) e^{-\beta s} ds, \tag{11}$$

(e.g., Caputo 2005, eq. 21 p. 86) with $Z(t) = \{E(t), H(t), A(t)\}$, and where T^* denotes optimal length of life and $U(*)$ denotes the maximized utility function (i.e., along the optimal paths for the controls, state functions, and for the optimal schooling age, retirement age, and length of life). Thus, for example, $q_E(t)$ represents the marginal value of remaining lifetime utility (from t onward) derived from additional skill capital $E(t)$. We refer to the co-state functions as the "marginal value of skill", the "marginal value of health", and the "marginal value of wealth" (these are also often referred to as the shadow prices of skill capital, of health capital, and of wealth).

Since skill capital $E(T)$ is unconstrained (free), the individual chooses it to have no value at the end of life, $q_E(T) = 0$. However, health capital $H(T)$ and assets $A(T)$ are constrained to their values $H_{\min}$ and A_T, respectively, and as a result they cannot be chosen not to have value at the end of life, and $q_H(T) \geq 0$ and $q_A(T) \geq 0$.

The transversality condition for the optimal length of schooling S, the optimal age of retirement R, and the optimal length of life T, follow from the dynamic envelope theorem (e.g., Caputo 2005, p. 293):

$$\begin{aligned}
\frac{\partial}{\partial S} \int_0^T \Im(t) dt &= \Im(S_-) - \Im(S_+) + \int_0^T \frac{\partial \Im(t)}{\partial S} dt = 0, && (12)\\
\frac{\partial}{\partial R} \int_0^T \Im(t) dt &= \Im(R_-) - \Im(R_+) + \int_0^T \frac{\partial \Im(t)}{\partial R} dt = 0, && (13)\\
\frac{\partial}{\partial T} \int_0^T \Im(t) dt &= \Im(T) = 0, && (14)
\end{aligned}$$

where S_-, R_- indicate the limit in which S, R are approached from below, and S_+, R_+ the limit in which S, R are approached from above. Conditions (12) and (13) have a natural interpretation in that the optimal length of schooling S and the optimal retirement age R are chosen such that there is no benefit of delaying entrance to the labor market (associated with optimal length of schooling S) and no benefit of delaying retirement R. $\Im(T)$ is the marginal value of life extension T (e.g., Theorem 9.1, p. 232 of Caputo, 2005), and the age at which life extension no longer has value defines the optimal length of life T^*.

2.2 Comparison with the literature

The canonical skill- and health-capital theories (Ben-Porath, 1967; Grossman, 1972a,b, 2000; Ehrlich and Chuma, 1990) models are sub-models of our formulation. The Ben-Porath (1967) model is obtained by removing the schooling decision S, the retirement decision R, leisure time $L(t)$, investment in health capital $I_H(t)$, $X_H(t)$ and $\tau_H(t)$, sick time $s[H(t)]$, and the dynamic equation (3) for health capital $H(t)$ from the model, and assuming fixed length of life T and a skill-capital production process of the form $f_E(t) = A\left[\tau_E(t)E(t)\right]^{\alpha}\left[X_E(t)\right]^{\beta}$ (Ben-Porath neutrality).[11]

The Grossman model is obtained by removing the schooling decision S, the retirement decision R, leisure time $L(t)$, investment in skill capital $I_E(t)$, $X_E(t)$ and $\tau_E(t)$, the dynamic equation (2) for skill capital $E(t)$, and the explicit condition for optimal length of life T (equation 14), and assuming a constant returns to scale health-production process: this consists of the standard assumption made in the health-capital literature of a linear health-production process $f_H(t) = I_H(t)$ and a Cobb-Douglas relation for health investment $I_H(t) = \mu_H(E)X_H(t)^{k_H}\tau_H(t)^{1-k_H}$. The efficiency of health investment $\mu_H(E)$ is allowed to be a function of exogenous skill capital E.

The formulations of Ehrlich and Chuma (1990) and Galama (2015) are as in Grossman (1972a,b; 2000) but assume a health-production process $f_H(t)$ with decreasing returns to scale in investment $I_H(t)$ and add an explicit condition for endogenous longevity (i.e. equation 14).

2.3 First-order conditions and interpretation

In this section we present and discuss the first-order (necessary) conditions and the transversality conditions of the optimal-control problem discussed above. The first-order conditions determine the optimal solutions of the controls skill-capital investment $I_E(t)$, health-capital investment $I_H(t)$, consumption $X_C(t)$, and leisure time $L(t)$, respectively.[12] Appendix A.2 provides detailed derivations. In this section we focus on working ages (i.e. the period between S and R). In section 2.4 we discuss the schooling and retirement phase.

Consumption and leisure The first-order conditions for consumption and leisure are standard

$$\frac{1}{q_A(t)}\frac{\partial U}{\partial X_C} = p_C(t)e^{\beta t}, \tag{15}$$

$$\frac{1}{q_A(t)}\frac{\partial U}{\partial L} = w[t, E(t)]e^{\beta t}. \tag{16}$$

[11]Even though the Ben-Porath model is formulated as maximizing lifetime earnings $Y(t)$, the characteristics of the model are very similar for a formulation in which the utility of lifetime consumption is maximized (as in this paper), with some exceptions (Graham, 1981).

[12]The first-order conditions for goods and services $X(t)$ are the same as for time inputs $\tau(t)$, as reflected in conditions (21) and (26) (see also Appendix A.2). As a result we have four rather than six controls.

Consumption $X_C(t)$ and leisure time $L(t)$ increase with current wealth $A(t)$ under the standard assumption of diminishing returns to wealth $\partial q_A(t)/\partial A(t) < 0$,[13] and with permanent income (the marginal value of wealth $q_A(t)$ decreases with permanent income).

Consumption and leisure decrease with their respective costs: the price of goods and services $p_C(t)$ (for consumption) and the opportunity cost of time $w[t, E(t)]$ (for leisure). Hence, anticipated increases in wages raise the opportunity cost of time and lead to a reduction in leisure (such a change occurs along the optimal lifecycle trajectory and does not affect the marginal value of wealth $q_A(t)$), but unanticipated (transitory or permanent) increases in wages also raise permanent income (such a change shifts the optimal life cycle trajectory by reducing the marginal value of wealth $q_A(t)$), and may therefore increase leisure if the permanent income effect dominates the opportunity cost of time effect.

Skill-capital investment The first-order condition for skill-capital investment $I_E(t)$ is given by

$$q_{e/a}(t) = \pi_E(t), \tag{18}$$

which equates the marginal benefit of skill-capital investment, given by the ratio of the marginal value of skill capital and the marginal value of wealth $q_{e/a}(t) \equiv q_E(t)/q_A(t)$, or the *relative marginal value of skill*, for short, to the marginal monetary cost of skill-capital investment $\pi_E(t)$.

The relative marginal value of skill is the solution to the dynamic co-state equation[14]

$$-\frac{\partial q_{e/a}}{\partial t} = \frac{\partial Y}{\partial E} + q_{e/a}(t)\left\{\frac{\partial f_E}{\partial E} - [d_E(t) + r]\right\} + q_{h/a}(t)\frac{\partial f_H}{\partial E}, \tag{20}$$

where the rate at which the relative marginal benefit of skill $q_{e/a}(t)$ depreciates over a short interval of time (left-hand side [LHS]) equals the sum of the direct benefits of skill capital and the contribution of skill capital to enhancing the value of the capital stocks (Dorfman, 1982). Skill capital contributes to wealth by raising earnings

[13] A natural and frequently made assumption is that financial capital (wealth) $A(t)$, skill capital $E(t)$, and health capital $H(t)$, increase remaining lifetime utility (from t onwards), but at a diminishing rate

$$\frac{\partial q_Z(t)}{\partial Z(t)} = \frac{\partial^2}{\partial Z(t)^2}\int_t^{T^*} U(*)e^{-\beta s}ds < 0, \tag{17}$$

with $Z(t) = \{E(t), H(t), A(t)\}$ (see 11).

[14] Or, alternatively

$$q_{e/a}(t) = \int_t^T e^{-\int_t^s\left[d_E[x]+r-\frac{\partial f_E}{\partial E}\right]dx}\left(\frac{\partial Y}{\partial E} + q_{h/a}(s)\frac{\partial f_H}{\partial E}\right)ds. \tag{19}$$

Thus the relative marginal value of skill $q_{e/a}(t)$ represents the sum of the lifetime production benefit (earnings) of skill $\partial Y/\partial E$ and the lifetime health-production benefit of skill $\partial f_H/\partial E$, discounted at the rate $d_E(t) + r - \partial f_E/\partial E$, where the discount rate is reduced as a result of the skill-production benefit of skill $\partial f_E/\partial E$.

$\partial Y/\partial E > 0$ (a production benefit), to skill by raising the efficiency of skill-capital production $\partial f_E/\partial E > 0$ (self-reinforcing self-productivity; valued at the relative marginal value of skill $q_{e/a}(t)$), and to health by raising the efficiency of health-capital production $\partial f_H/\partial E > 0$ (cross-fertilizing self-productivity; valued at the relative marginal value of health $q_{h/a}(t)$). The relative marginal value of skill appreciates with $d_E(t)$ (biological aging depletes the stock of skill, a cost) and the rate of return to capital r (the opportunity cost of investing in skill capital rather than in the stock market), where both costs are valued at the relative marginal value of skill $q_{e/a}(t)$.

The marginal cost of skill-capital investment is defined as

$$\pi_E(t) \equiv \frac{p_E(t)}{\frac{\partial f_E}{\partial I_E}\frac{\partial I_E}{\partial X_E}} = \frac{w[t, E(t)]}{\frac{\partial f_E}{\partial I_E}\frac{\partial I_E}{\partial \tau_E}}. \tag{21}$$

The marginal cost of investment in skill capital increases with the price of investment goods and services $p_E(t)$, and the opportunity cost of not working $w[t, E(t)]$, and decreases in the efficiency of the use of investment inputs in the skill production process, $\partial f_E/\partial X_E$ and $\partial f_E/\partial \tau_E$. Because of diminishing returns to scale in skill-capital investment, the marginal cost of skill capital is an increasing function of the level of investment goods / services $X_E(t)$ and investment time inputs $\tau_E(t)$, and hence an increasing function of the level of investment $I_E(t)$ (see 21).[15] Intuitively, due to concavity in investment $I_E(t)$ of the skill production process $f_E(t)$, the higher the level of investment, the smaller the improvement in skill $E(t)$. As a result, the effective cost of investment is higher.

In sum, the decision to invest in skill today (18) weighs the current monetary price and current opportunity cost of time (see 21) with its future benefits (from t to T), consisting of increased earnings, and more efficient skill and health production (see 19).

Health-capital investment Analogous to skill-capital investment, the first-order condition for health-capital investment is given by

$$q_{h/a}(t) \quad = \quad \pi_H(t), \tag{23}$$

where the marginal benefit of health investment $q_{h/a}(t)$ equals the ratio of the marginal value of health and the marginal value of wealth $q_{h/a}(t) \equiv q_H(t)/q_A(t)$, or the *relative marginal value of health*, for short, and $\pi_H(t)$ represents the marginal monetary cost of health-capital investment.

[15]Because of concavity of f_E the first derivatives of the production process $\partial f_E/\partial X_E$ and $\partial f_E/\partial \tau_E$, are monotonically decreasing functions of X_E and τ_E, respectively. For example, for the functional form $f_E[I_E(t), E(t), H(t)] \equiv f_E^*[E(t), H(t)]I_E(t)^{\alpha_E}$ (where $0 < \alpha_E < 1$ [diminishing returns]) and a Cobb-Douglass relation between the inputs X_E, τ_E and the output investment $I_E(t)$, $I_E(t) \equiv X_E(t)^{k_E}\tau_E(t)^{1-k_E}$, we have

$$\pi_E(t) = \frac{p_E(t)^{k_E} w[t, E(t)]^{1-k_E}}{\alpha_E f_E^*[E(t), H(t)] k_E^{k_E}(1-k_E)^{1-k_E}} I_E(t)^{1-\alpha_E} \equiv \pi_E^*(t) I_E(t)^{1-\alpha_E}. \tag{22}$$

The relative marginal value of health is the solution to the co-state equation[16]

$$
\begin{aligned}
-\frac{\partial q_{h/a}}{\partial t} &= \frac{1}{q_A(t)}\frac{\partial U}{\partial H}e^{-\beta t} + \frac{\partial Y}{\partial H} + q_{e/a}(t)\frac{\partial f_E}{\partial H} \\
&+ q_{h/a}(t)\left\{\frac{\partial f_H}{\partial H} - [d_H(t)+r]\right\} + \frac{\lambda_{H_{\min}}(t)}{q_A(t)},
\end{aligned} \tag{25}
$$

and the marginal cost of health investment is defined as

$$
\pi_H(t) \equiv \frac{p_H(t)}{\frac{\partial f_H}{\partial I_H}\frac{\partial I_H}{\partial X_H}} = \frac{w[t,E(t)]}{\frac{\partial f_H}{\partial I_H}\frac{\partial I_H}{\partial \tau_H}}. \tag{26}
$$

Like skill capital, the benefits of health capital consist of increasing earnings $\partial Y/\partial H > 0$, and raising the efficiency of skill production $\partial f_E/\partial H > 0$ (valued at the relative marginal value of skill $q_{e/a}(t)$). Here too the relative marginal value of health appreciates with $d_H(t)$ and the rate of return to capital r, where both costs are valued at the relative marginal value of health $q_{h/a}(t)$. Unlike skill capital, health also has a consumption benefit (direct utility) $\partial U/\partial H$, health enables life extension (see 14; Ehrlich and Chuma, 1990), and it is unclear whether health enhances or reduces the efficiency of health production $\partial f_H/\partial H$. Further, the constraint that health cannot fall below a minimum level $H_{\min}$ is reflected in an additional term $\lambda_{\mathrm{H_{min}}}(t)/q_A(t)$ in (25). The term is absent from the relation for the relative marginal value of skill (20), and it increases the rate at which the marginal value of health depreciates. In practice, employing the condition entails restricting solutions to those where the constraint is not imposing.

The marginal cost $\pi_H(t)$ of health investment increases with the price of goods and services in the market $p_H(t)$, the opportunity cost of time $w[t; E(t)]$, and in the level of investment $I_H(t)$ due to decreasing returns to scale. It decreases in the efficiency of the use of investment inputs in the health production process, $\partial f_H/\partial X_H$ and $\partial f_H/\partial \tau_H$.

In sum, similar to investment in skill, the decision to invest in health today (23) weighs the current monetary price and current opportunity cost of time (see 26) with its future benefits (from t to T), consisting of enhanced utility, increased earnings, more efficient skill production, and a longer life (see 24).

Dynamics The dynamic equations for skill (2) and for the relative marginal value of skill (20), together with the initial, end, and transversality conditions, determine the evolution of skill and skill investment over the lifecycle. Likewise, the dynamic equations for health (3) and for the relative marginal value of health (25) determine the evolution of health and

[16] Or, alternatively

$$
\begin{aligned}
q_{h/a}(t) &= q_{h/a}(T)e^{-\int_t^T\left[d_H(x)+r-\frac{\partial f_H}{\partial H}\right]dx} \\
&+ \int_t^T e^{-\int_t^s\left[d_H[x]+r-\frac{\partial f_H}{\partial H}\right]dx}\left(\frac{1}{q_A(0)}\frac{\partial U}{\partial H}e^{-(\beta-r)s} + \frac{\partial Y}{\partial H} + q_{e/a}(s)\frac{\partial f_E}{\partial H} + \frac{\lambda_{H_{\min}}(s)}{q_A(s)}\right)ds
\end{aligned} \tag{24}
$$

health investment.[17] Skill capital and health capital have different initial and terminal conditions. Individuals begin life with limited skills and end life with various degrees of cognitive and mental fitness. This notion is captured in the skill-capital literature by an initial low level of skill E_0 and an end value $E(T)$ that is, apart from being non-negative, unconstrained. Because there is no restriction on the terminal value of the stock of skill $E(T)$, it is chosen such that skill no longer has value at the end of life, $q_E(T) = 0$ (see Heckman, 1976; Chiang, 1992). Thus the relative marginal value of skill capital $q_{e/a}(t)$, and therefore investment in skill, decreases over the life-cycle and approaches zero at the end of life (see 19).

In contrast, most people start adult life with a healthy body, and for natural causes of death the terminal state of health is universally frail. The notion that health cannot be sustained below a certain minimum level is captured in the health-capital literature by the condition $H(T) = H_{\min}$. Health capital eventually decreases over the lifecycle and because the terminal health stock is restricted to $H_{\min}$, it cannot be chosen to have no value, $q_H(T) \geq 0$.

Conjecture 1: Skill capital generally grows as a result of investment but health capital eventually declines.

Conjecture 2: Skill is valued early while health is valued later in life.

Limiting the discussion to adulthood, skill capital is found to increase, at least initially (e.g., Becker 1964, Ben-Porath, 1967), while health capital is found to decrease with age (e.g., Grossman, 1972a;b). Skill-capital investment is thus characterized by a production process that enables improvements in the stock of skill, while health-capital investment is characterized by a production process that (eventually) cannot prevent declining health, no matter how much one invests in it (a dismal fact of life; conjecture 1).

Further, the empirical and theoretical literatures suggest that investments in skill capital tend to decrease with age (e.g., Becker, 1964), while investments in health tend to increase with age (e.g., Zweifel, Felder, and Meiers, 1999). This suggests that the relative marginal value of health $q_{h/a}(t)$ increases with age, while the relative marginal value of skill $q_{e/a}(t)$ decreases with age. Skill is valued early while health is valued later in life (conjecture 2).

[17] The evolution of assets is given by (4), and the marginal value of assets is determined by its co-state equation (see equation 37 in Appendix A.2):

$$-\frac{\partial q_A}{\partial t} = q_A(t)r. \tag{27}$$

2.4 The schooling, work, and retirement decision

The transition from school to work In school, the opportunity cost of time investments (e.g., attending class, studying, completing assignments) is low since students do not work and the time that would otherwise be spent working can now be devoted to skill investment $\tau_E(t)$, health investment $\tau_H(t)$, and leisure $L(t)$.[18] Though not exclusively, individuals will use the schooling period (i.e. $t < S$) as a period of life to invest in skill capital $E(t)$, since skill is valued most early in life (see conjecture 2), and because during the schooling phase individuals are encouraged to invest in skill through an education transfer $b_S(t)$, and/or through subsidized schooling (reduced $p_E(t)$; e.g., public schooling).

As individuals gain skill their potential labor earnings increase, and at some point it becomes attractive to join the labor market. Individuals will join the labor market at the age S, the age at which the benefits of work exceed the benefits of staying in school (see 12). Noting that state and co-state functions are continuous in S, and $\lambda_{\tau_w}(S)w[t,E(t)]\tau_w(S) = 0$, the transversality condition (12) reduces to

$$\begin{aligned} Y(S_+) &= b_S(S) + \frac{1}{q_A(S)}\left[U(S_-) - U(S_+)\right]e^{-\beta S} \\ &+ q_{e/a}(S)\left[f_E(S_-) - f_E(S_+)\right] + q_{h/a}(S)\left[f_H(S_-) - f_H(S_+)\right] \\ &+ p_H(S)\left[X_H(S_+) - X_H(S_-)\right] + p_E(S_+)X_E(S_+) - p_E(S_-)X_E(S_-) \\ &+ p_C(S)\left[X_C(S_+) - X_C(S_-)\right], \end{aligned} \tag{28}$$

where $Y(S_+) = w(S)\tau_w(S_+)$, and we have replaced the limits S_- and S_+ with S for functions that are continuous in S. The LHS of (28) represents the benefits of entering the labor market consisting of labor income $Y(S_+)$. The right-hand side (RHS) represents the benefit of staying in school, consisting of the schooling subsidy $b_S(S)$ (first term), the monetary value of utility from more leisure time (second term),[19] and the value of higher levels of skill investment and health investment while in school due to the lower opportunity cost of time (third and fourth term). Further, if time substitutes for goods and services $X_H(t)$ in the production of health investment, then the fifth term on the RHS represents benefits of schooling in terms of reduced expenditures. The sixth

[18] In the absence of earnings from wages, the opportunity cost of time is not determined by the wage rate $w(t)$ but by the constraint (9) that individuals not work $\tau_w(t) = 0$. A simple heuristic argument can be made that the opportunity cost of time is always lower at every age during schooling years (and retirement years), as follows. When individuals are allowed to work they may devote very little time to work, but they will never choose not to work since the decision to work provides an additional margin of adjustment with some benefit. Thus the total time available that can be devoted to leisure, consumption and investment is larger when not working, and thus the opportunity cost of time lower. A comparison of the first-order conditions in Appendix A.2 shows that one can obtain the first-order conditions for non-working ages simply by replacing all occurances of the monetary value of the opportunity cost $q_A(t)w(t;E)$ with $\lambda_{\tau_w}(t)w(t;E)$.

[19] Even if leisure and consumption are substitutes in utility, utility right before the transition from schooling to work is arguably still higher, $U(S_-) - U(S_+) > 0$, as the effect of a change in consumption on utility is a second-order effect (and thus relatively small), operating through the effect that a change in leisure time has on the marginal utility of consumption.

term reflects the possibility that the cost of schooling $p_E(t)X_E(t)$ (for $t < S$) may be subsidized, providing another benefit of staying in school. The last term reflects changes in consumption as a result of changes in the marginal utility of consumption due to reduced leisure time while working.

The transition from work to retirement After graduating from school, individuals enter the labor market, and start working. Time previously devoted to skill investment, health investment, and leisure is reduced. As a result, skill increases at a slower pace, and health deteriorates faster. Declining health reduces earnings – through increased sick time and by increasing time devoted to health investment –, and retirement becomes increasingly attractive. Retirement is in part attractive because it lowers the opportunity cost of time. The time otherwise spent working can then be devoted to leisure $L(t)$, and time inputs into health-capital investment $\tau_H(t)$, and skill-capital investment $\tau_E(t)$.

The optimal retirement age is determined by the transversality condition (13, see also Kuhn et al. 2012). Noting that state and co-state functions are continuous in R, and $\lambda_{\tau_w}(R)\tau_w(R) = 0$, the transversality condition (13) reduces to

$$\begin{aligned}
Y(R_-) &= \left\{ b(R) - \frac{\partial b(R)}{\partial R}\frac{1}{r}\left[1 - e^{-r(T-R)}\right]\right\} \\
&+ \frac{1}{q_A(R)}\left[U(R_+) - U(R_-)\right]e^{-\beta R} \\
&+ q_{e/a}(R)\left[f_E(R_+) - f_E(R_-)\right] + q_{h/a}(R)\left[f_H(R_+) - f_H(R_-)\right] \\
&+ p_H(R)\left[X_H(R_-) - X_H(R_+)\right] + p_E(R)\left[X_E(R_-) - X_E(R_+)\right] \\
&+ p_C(R)\left[X_C(R_+) - X_C(R_-)\right], \qquad (29)
\end{aligned}$$

where $Y(R_-) = w(R)\tau_w(R_-)$. The optimal age of retirement R requires the benefits of working, consisting of labor income $Y(R_-)$, to equal the benefits of retirement.

Intuitively, if utility $U(t)$, consumption, and investments in skill and health capital were continuous in R, and the state pension annuity $b(R) = b_R$ were independent of the age of retirement, the decision to retire would simply be determined by the age R at which earnings in retirement b_R exceeded, for the first time, earnings from work $Y(t)$. Generous retirement (e.g., social security) benefits b_R and low labor income $Y(t)$ (e.g., due to worsening health and declining skill capital with age) encourage early retirement.

In practice, the pension benefit $b(R)$ is a function of the age of retirement R, typically at least initially increasing in R. It is then attractive for individuals to delay retirement in order to receive higher benefits $b(R)$ per period. However, this comes at the cost of a shortened horizon $T - R$ over which these benefits are received, as reflected in the term $(\partial b(R)/\partial R)(1/r)\left[1 - e^{-r(T-R)}\right]$. Further, individuals value the utility from additional leisure in retirement (second term on the RHS), they value the additional investment in skill capital and health capital due to the lower opportunity cost of time (third and fourth term on the RHS), and there are potential reductions in expenditures on consumption and skill and health-capital investment goods and services (terms five, six, and seven on the

RHS). As a result, individuals do not need to be compensated dollar for dollar in income, and retirement occurs while pension benefits are less than labor income.

3 Model predictions

In this section we summarize results, analyze the dynamics of the model, and make predictions. In section 3.1 we discuss life-cycle trajectories, and in section 3.2 we present comparative dynamic analyses to explore cross-sectional heterogeneity in these profiles.

3.1 Lifecycle trajectories

The characteristics of the solutions are visually represented in Figure 3, where S is years of schooling, R is the age of retirement, and T denotes total lifetime.[20] The top panel presents the life-cycle profile of skill investment $I_E(t)$ (left), and skill $E(t)$ (right). Since skill determines wages $w(t, E)$, they show a similar pattern (for this reason wages are not separately shown). The bottom panel presents the life-cycle profile of health investment $I_H(t)$ (left) and health $H(t)$ (right).

The various benefits of skill investment (in the production of earnings, skill and health) are high early in life as the horizon over which benefits can be accrued is still long. Moreover, low levels of skill early in life increase the incentives to invest in skill due to diminishing returns to skill (see 18, 20, and 21). During schooling, the use of time inputs in skill investment is encouraged, as the opportunity cost of time is low when one is not allowed to work (i.e., mimimum schooling ages reduce the cost of time). Further, individuals potentially receive a transfer $b_S(t)$ and/or schooling is subsidized, i.e. small $p_E(t)$. This further encourages investment. Thus skill investment is high, and skill increases rapidly, early in life (top-left and top-right panel) and, in particular, during the schooling phase.

As skill grows, the benefit of work (earnings) increases, individuals leave school and start working. Less time will be devoted to skill investment because of the higher opportunity cost of time (hence the drop in the level of investment $I_E(t)$ at $t = S$, top-left panel), and the rate at which skill is produced slows (hence the downward change in the slope for $E(t)$ at $t = S$, top-right panel).[21] Skill $E(t)$ (top-right panel) may eventually decline as biological deterioration outweighs declining skill-capital investment (see 2).

After retirement, time spent working is zero. The greater availability of time encourages individuals to invest more time in skill, hence the jump upward in skill

[20]Note that these are based on analytical reasoning, not on a numerical simulation.

[21]The jump in skill investment $I_E(t)$ is due to the increased opportunity cost of time. As Becker (1964, Chapter 3) argues, even though on-the-job training may simply happen while on the job, there is still an opportunity cost of time for the worker as firms are not willing to pay for perfectly general training (as opposed to firm-specific training). There is also a change in slope. For illustrative purposes, $I_E(t)$ is shown as decreasing more rapidly during working life, but the opposite is possible too.

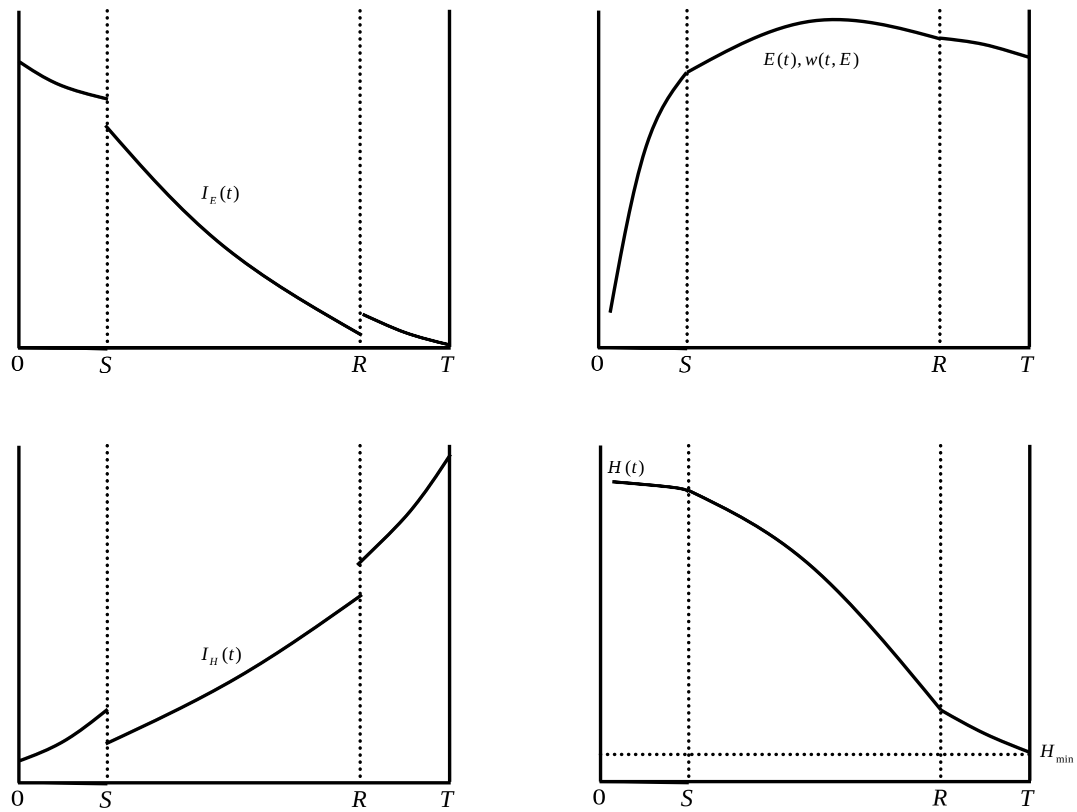

Figure 3: *Illustration of the time paths for skill investment $I_E(t)$ (top left), skill capital $E(t)$ and the wage rate $w[t, E(t)]$ (top right), health investment $I_H(t)$ (bottom left), and health $H(t)$ (bottom right)*

investment $I_E(t)$[22] (top-left) and the upward change in the slope of the skill $E(t)$ profile (top-right) at $t = R$. While skill is no longer useful in the production of earnings $\partial Y/\partial E = 0$, during retirement it still delivers an important health-production benefit $\partial f_H/\partial E > 0$. Thus, individuals will continue to invest in skill capital after retirement. Note that this finding is in sharp contrast to the skill-capital literature (e.g., Becker, 1964; Ben-Porath, 1967), which predicts no skill investment after retirement.[23] Eventually,

[22]This result is somewhat counterintuitive since after retirement there is no longer a production benefit of skill, $\partial Y/\partial E = 0$ for $t > R$, and so one might view skill as less valuable. The co-state variables $q_E(t)$ and $q_A(t)$, however, are continuous at $t = R$, so there is no discontinuity in the relative marginal value of skill $q_{e/a}(t)$. Given the equilibrium condition $q_{e/a}(R) = \pi_E(R)$ (see 18), and since the cost of time is reduced, there is a discontinuous increase in investment $I_E(t)$ at $t = R$ (see 21).

[23]Some elderly enroll in education programs during retirement. Plausibly, skill provides, besides the health benefit, additional benefits, such as a home-production benefit (e.g., cognition enables individuals to remain independent) or a consumption benefit (individuals enjoy learning). Those are not part of our theory, but can be readily included.

investment in skill capital declines to zero (top-left panel), since individuals place no marginal value on the terminal stock of skill $q_E(T) = 0$ (see section 2.1).

At early stages in the life cycle, the individual is endowed with a large stock of health $H(t)$. As a result, the benefit of health investment is relatively low due to diminishing returns in health, and it may be optimal to devote resources to skill capital instead (low health investment $I_H(t)$, bottom-left panel). As the individual ages, the stock of health declines monotonically (bottom-right panel).[24] Health investments then become essential to counteract biological aging and to extend life. As a result, health investment, in contrast to skill investment but in line with empirical evidence (Zweifel, Felder, and Meiers, 1999; De Nardi, French and Jones, 2010), increases over the life cycle (bottom-right panel).[25] Retirement further encourages health investment due to the reduced cost of time inputs. Combining this with the earlier discussion for skill, we conclude that, as one would expect, the schooling period is primarily used to invest in skill, and the retirement period is primarily used to invest in health.

Like skill capital, health capital does not contribute to earnings during retirement and therefore the production benefit $\partial Y/\partial H$ is zero. However, health still provides an important home-production benefit as better health reduces sick time, time that can be devoted to leisure $L(t)$, investment in skill $\tau_E(t)$, and investment in health $\tau_H(t)$. Unlike skill capital, health capital also provides direct utility in retirement, providing additional incentives to invest in health after retirement. The health stock eventually reaches a minimum level $H_{\min}$ at the end of life T (indicated by the dotted horizontal line).

3.2 Cross-sectional variation in the life-cycle trajectories

The life-cycle trajectories discussed in section 3.1 can be viewed as representing the average individual in a representative sample. We are also interested in understanding cross-sectional heterogeneity in these profiles. Our theory describes the entire lifecycle, and is dynamic, limiting the use of comparative static analyses. To gain further insight into the characteristics of the theory, we have to resort to comparative dynamic analyses, which allow analyzing variation in the lifecycle profiles with respect to the three types of resources an individual possesses, financial capital (wealth), skill capital, and health capital, as well as other model parameters of interest.

Following Ben-Porath (1967) and Heckman (1976) we can make some convenient assumptions to arrive at a tractable version of our general theory that permits derivation of analytical expressions for the comparative dynamic results. The simpler model retains the essential characteristics of the general theory. There are some costs associated

[24] Similar to skill, the rate of health decline changes at S and at R due to changes in the opportunity cost of time, and associated changes in the level of health investment.

[25] In part, this is because the terminal level of health is constrained to $H_{\min}$. As a result, the relative marginal value of health at the end of life $q_{h/a}(T)$ does not have to be zero. In contrast, the end condition for skill, $E(T)$ free, implies $q_E(T) = 0$ and hence $I_E(T) = 0$ so that skill investment eventually has to decline (see the discussion in section 2.3). Thus, a crucial difference between skill and health is the notion that individuals end life in universally poor health but with varying levels of skill.

with the simplifications, which we discuss in detail in Appendix section A.4, but the benefits arguably outweigh the costs. Most importantly, the assumptions enable obtaining analytical expressions for the comparative dynamic analyses. We find that the predictions of the simpler model also hold for the general model with some nuanced differences (which are discussed in detail in Appendix A.6). Since our approach does not solely rely on the simpler model we obtain robust comparative dynamic results.

We start by introducing the simplified theory.

3.2.1 A simpler tractable model

Individuals maximize a constant relative risk aversion (CRRA) lifetime utility function

$$U(t) = \frac{1}{1-\rho}\left(X_C(t)^{\zeta}\left\{L(t)[E(t)+H(t)]\right\}^{1-\zeta}\right)^{1-\rho}, \tag{30}$$

with ζ the "share" of consumption and $1-\zeta$ the "share" of leisure in utility, and $1/\rho$ the elasticity of substitution. Consumption $X_C(t)$ and "effective" leisure time $L(t)[E(t)+H(t)]$ are complements in utility if $\rho < 1$ and substitutes in utility for $\rho > 1$. Leisure time $L(t)$ is multiplied by $E(t)+H(t)$, reflecting the notion that human capital (consisting of the sum of skill and health capital, $E(t)+H(t)$) augments the agent's consumption time (Heckman, 1976). The utility function is maximized subject to the same dynamic constraints for skill capital (2), for health capital (3), and for assets (4), as in the general framework.

We assume no sick time $s[H(t)]$, that earnings consist of the product of human capital, $E(t)+H(t)$, and the fraction of time available for work

$$Y[E(t), H(t)] \quad = \quad [E(t)+H(t)]\left[1-\tau_E(t)-\tau_H(t)-L(t)\right], \tag{31}$$

and, last, that the production functions of skill capital and of health capital are of a Cobb-Douglas form,

$$\begin{aligned} f_E[\tau_E(t), X_E(t), E(t), H(t)] &= \theta_E(t)\left\{\tau_E(t)\left[E(t)+H(t)\right]\right\}^{\alpha_E} X_E^{\beta_E}, \\ &= \mu_E(t) q_{e/a}(t)^{\frac{\gamma_E}{1-\gamma_E}}, \qquad (32) \\ f_H[\tau_H(t), X_H(t), E(t), H(t)] &= \theta_H(t)\left\{\tau_H(t)\left[E(t)+H(t)\right]\right\}^{\alpha_H} X_H^{\beta_H}, \\ &= \mu_H(t) q_{h/a}(t)^{\frac{\gamma_H}{1-\gamma_H}}, \qquad (33) \end{aligned}$$

where $\theta_E(t)$ and $\theta_H(t)$ denote the technologies of production of skill investments and health investments, respectively, $\gamma_E = \alpha_E + \beta_E < 1$, and $\gamma_H = \alpha_H + \beta_H < 1$ (diminishing returns to scale).[26] The functions $\mu_E(t)$ and $\mu_H(t)$ are generalized productivity factors

$$\mu_E(t) \equiv \left[\frac{\alpha_E^{\alpha_E}\beta_E^{\beta_E}\theta_E(t)}{p_E(t)^{\beta_E}}\right]^{\frac{1}{1-\gamma_E}}, \tag{34}$$

$$\mu_H(t) \equiv \left[\frac{\alpha_H^{\alpha_H}\beta_H^{\beta_H}\theta_H(t)}{p_H(t)^{\beta_H}}\right]^{\frac{1}{1-\gamma_H}}. \tag{35}$$

The technologies of production $\theta_E(t)$, $\theta_H(t)$, and the generalized productivity factors $\mu_E(t)$, $\mu_H(t)$, can be considered as being determined by technology as well as biology.

The begin and end conditions H_0, $H(T) = H_{\min}$, E_0, A_0, $A(T) = A_T$, and the transversality conditions $q_E(T) = 0$, and $\Im(T) = 0$, also apply here. The analytical solutions of the simpler model are presented in Appendix A.3.

3.2.2 Comparative dynamics

Comparative dynamic analyses allow us to analyze differences in behavior as a function of model parameters. We start with an analysis of endowed wealth, health, and skill.

Consider a generic control, state, or co-state function $g(t)$, and a generic variation δZ_0 in an initial condition or model parameter. The effect of the variation δZ_0 on the optimal path of $g(t)$ can be broken down into variation for fixed longevity T and variation due to the resulting change in the horizon T

$$\frac{\partial g(t)}{\partial Z_0} = \left.\frac{\partial g(t)}{\partial Z_0}\right|_T + \left.\frac{\partial g(t)}{\partial T}\right|_{Z_0}\frac{\partial T}{\partial Z_0}. \tag{36}$$

The comparative dynamic effects of a small perturbation in initial wealth δA_0, initial skill δE_0, and initial health δH_0 are summarized in Table 1.[27] Detailed derivations are provided in Appendix section A.5.[28]

We distinguish between two cases, one in which length of life is fixed (exogenous), and one in which length of life can be freely chosen (endogenous).

Generalized Heckman result: Absent ability to increase the horizon over which benefits can be accrued (fixed length of life T), additional wealth does not lead to more skill investment and health investment, leaving skill and health unchanged (rows 1 to 8 for T fixed). The additional wealth is solely used to finance additional consumption and

[26] Proof that the skill f_E and health f_H production functions can be expressed in terms of the relative marginal value of skill $q_{e/a}(t)$, and of health $q_{h/a}(t)$, is provided in Appendix A.3 (see equations 54 to 57).

[27] Note that we can restart the problem at any time t, taking $A(t)$, $E(t)$, and $H(t)$, as the new initial conditions. Thus the comparative dynamic results derived for variation in initial wealth δA_0, initial skill δE_0, and initial health δH_0, have greater validity, applying to variation in wealth $\delta A(t)$, skill $\delta E(t)$, and health $\delta H(t)$, at any time $t \in [0, T)$.

[28] See equations (84), (85), and (86) for initial wealth A_0, equations (87) to (91) for initial skill E_0, and equations (93) to (97) for initial health H_0.

Table 1: Comparative dynamic effects of initial wealth A_0, initial skill E_0, and initial health H_0, on the state and co-state functions, control functions and the parameter T.

	δA_0		δE_0		δH_0	
Function	T fixed	T free	T fixed	T free	T fixed	T free
$E(t)$	0	>0	>0	>0	0	>0
$q_{e/a}(t)$	0	>0	0	>0	0	>0
$X_E(t)$	0	>0	0	>0	0	>0
$\tau_E(t)\,[E(t)+H(t)]$	0	>0	0	>0	0	>0
$H(t)$	0	>0	0	>0	≥ 0	>0
$q_{h/a}(t)$	0	>0	0	>0	<0	+/-
$X_H(t)$	0	>0	0	>0	<0	+/-
$\tau_H(t)\,[E(t)+H(t)]$	0	>0	0	>0	<0	+/-
$A(t)$	≥ 0	+/-	+/-	+/-	+/-	+/-
$q_A(0)$	<0	$<0^\dagger$	<0	$<0^\dagger$	<0	$<0^\dagger$
$X_C(t)$	>0	$>0^\dagger$	>0	$>0^\dagger$	>0	$>0^\dagger$
$L(t)\,[E(t)+H(t)]$	>0	$>0^\dagger$	>0	$>0^\dagger$	>0	$>0^\dagger$
T	n/a	>0	n/a	>0	n/a	>0

Notes: 0 is used to denote 'not affected', $+/-$ is used to denote that the sign is 'undetermined', n/a stands for 'not applicable', and † is used to denote that 'the sign holds under the plausible assumption that the wealth effect dominates the effect of life extension'. This is consistent with the empirical finding (Imbens, Rubin and Sacerdote, 2001; Juster et al. 2006; Brown, Coile, and Weisbenner, 2010) that additional wealth leads to higher consumption, even though the horizon over which consumption takes place is extended (see section A.5 for further detail).

leisure (rows 11 and 12). Both skill capital and health capital are forms of wealth, in the sense that they increase wages and therefore lifetime wealth (reducing the marginal value of initial wealth $q_A(0)$). Thus a positive variation in skill δE_0 and in health δH_0 operates in a manner similar to a positive variation in wealth δA_0 (see columns 3 and 5 in the Table, 88 and 93), with some differences: endowed skill E_0 leads to greater skill, endowed health H_0 leads to greater health, and endowed health H_0 reduces the relative marginal value of health $q_{h/a}(t)$ and thereby the demand for health investment. Importantly, also for additional skill and additional health there are no additional investments made, and for additional health, health investments are even reduced. Thus, absent ability to extend life T, or in other words without any inequality in life expectancy, associations between wealth, skill and health are absent or small.

This lack of an association between skill and wealth (and in our case also health) in the Ben-Porath model has been noted before (Heckman, 1976).[29] It arises because length of life (fixed in the traditional human-capital literature) is a crucial determinant of the

[29] Levhari and Weiss (1974) also note the problem in a simple two-period human-capital model with uncertainty but do not explicitly derive the result. Graham (1981) suggests the lack of an association is due to the fact that in the Ben-Porath model individuals maximize lifetime earnings, and not utility. We find, however, that it also holds for a model in which individuals maximize lifetime utility.

return to investments. The intuition is straightforward for health investment. For fixed length of life, in both the general and simpler model, any additional health investment needs to be compensated by eventual lower investment in order for health to reach $H_{\min}$ at $t = T$. The response to additional resources is therefore muted. As a result, there are no strong associations between wealth, skill, and health for fixed T, and $\partial g(t)/\partial Z_0|_T$, the first term on the RHS of (36), is generally small for variation δZ_0 in any model parameter of interest. This result also holds for the general model (see Appendix A.6 for detail).

Generalized Ehrlich and Chuma result: Now consider the case where T is free. The bottom row of Table 1 shows that positive variations in endowments, in the form of wealth, skill or health, lead to a longer life span. For variation in initial wealth A_0, the intuition is as follows. At high values of wealth (and hence consumption and leisure), additional consumption or leisure per period yields only limited utility due to diminishing marginal utility of consumption and leisure. In contrast, investments in health extend life, increasing the period over which they can enjoy the utility benefits of leisure and consumption. With sufficient wealth one starts caring more about other goods, in particular health. Hence, wealth increases health investment and thereby health, $\partial H(t)/\partial A_0 > 0 \quad \forall t$, and extends life $\partial T/\partial A_0 > 0$ (Ehrlich and Chuma, 1990). Endowments in skill and health are also forms of wealth, in the sense that skill and health increase earnings and therefore lifetime wealth. Hence, a similar reasoning can be applied to variations in initial skill and initial health. Thus, wealthy, skilled, and healthy individuals live longer. This prediction also holds for the general model (see Appendix A.7 for detail).

Thus, in our substantially richer model, Heckman's (1976) result also holds for health: for fixed (exogenous) length of life T there are no associations between wealth, skill and health. In doing so, we generalize Heckman's result. In our richer model individuals with greater resources also live longer. In doing so we generalize Ehrlich and Chuma's (1990) result. In what follows we will refer to these findings as the generalized Heckman and generalized Ehrlich and Chuma results. They are reassuring but not entirely novel results. Of greater interest are the following three novel predictions of our theory.

Prediction 1: Health improves skill formation beyond the schooling period.

The last column of Table 1 shows that healthier individuals invest more in skill, and are more skilled as a result. This effect of health on skill formation is absent from the traditional skill-capital literature, where health does not feature as a separate capital stock. It is also absent from the health-capital literature, where skill is treated as determined outside the model. A literature examines the effect of health on schooling and educational attainment (see Bleakley, 2010a for a review), but, to the best of our knowledge, no studies exist on the effect of health on skill formation beyond the schooling age.

The theory highlights three mechanisms underlying this effect. First, healthier people live longer, which increases the returns to skill investment (see 83, 96, and 97). Second, health raises the productivity of future skill investments through cross-fertilizing dynamic complementarity (see 2 and 32). Finally, in the general model, health and skill are complementary in generating earnings, since earnings are the product of wages (predominantly influenced by skill) and time spent working (predominantly affected by health). Hence, health raises the return to investment in skill because it increases the amount of time available for work. See (6) and Appendix A.8 for a detailed discussion.

Prediction 2: Longevity gains generate health and economic inequality. That is, if additional resources enable life extension, associations between wealth, skill, health, and technology are stronger.

Gains in life expectancy play a powerful role in generating associations between wealth, skill, health, and technology. Equation (36) illustrates this. From our generalized Heckman result we obtain that the first term on the RHS $\partial g(t)/\partial Z_0|_T$ is generally small for $g(t) = \{E(t), H(t)\}$, and for variation δZ_0 in any model parameter of interest: additions (or reductions) in resources do not change investment in skill and health much for fixed (exogenous) T. Thus, the size of the effect of Z_0 on $g(t)$ depends on the sign of $\partial g(t)/\partial T|_{Z_0}$ and increases with the degree of life extension $\partial T/\partial Z_0$. From our generalized Ehrlich and Chuma result, we have $\partial T/\partial Z_0 > 0$ for $Z_0 = \{A_0, E_0, H_0, \mu_E(t), \mu_H(t)\}$, since wealthy, skilled and healthy individuals live longer. Finally, we have $\partial g(t)/\partial T|_{Z_0} > 0$ for $g(t) = \{E(t), H(t)\}$ (see 83).[30]

If resources, biology, medical technology, institutional, environmental and/or other factors, do not allow for life extension ($\partial T/\partial Z_0$ small), then the effect closely resembles that of the fixed T case. As in the fixed T case, there is a small association between wealth, skill, health, and technology (small $\partial g(t)/\partial Z_0$). In contrast, if additional resources afford considerable life extension ($\partial T/\partial Z_0$ large), the horizon over which the benefits of skill investment and health investments can be reaped is larger. Further, utility from leisure and consumption can be enjoyed with additional years of life. Together, these various benefits of life extension substantially raise investment in skill and in health, thereby improving skill and health. The prediction also holds for the general model, with some differences (see Appendix A.9 for detail).

An imperfect yet empirically tractable measure of the extent to which resources enable life extension $\partial T/\partial Z_0$ is inequality in life expectancy T. One would expect countries with larger inequality in life expectancy, to have larger inequality in, e.g., education. While this does not provide conclusive evidence, Figure 1 shows exactly such a pattern, with a statistically significant positive association between inequality in life expectancy and inequality in education.[31]

[30] The sign for $\partial A(t)/\partial T|_{Z_0}$ is ambiguous because the additional resources (endowments in wealth, skill, or health, or technological improvement) have to be spread over a longer horizon, but the longer horizon at the same time encourages greater investment in skill and in health, which in turn accumulates wealth.

[31] Clearly, causality could operate in either direction, and there are a number of third factors, including

Prediction 3: Complementarity effects reinforce associations between wealth, skill, health, and technology. For example, life expectancy and skill-capital productivity reinforce each other in generating skill.

The analytical comparative dynamic expressions of the simpler model can be employed to not only study the sign of the comparative dynamic effects, but also to study complementarities between model parameters. Is, for example, the effect of endowed skill on health formation $\partial H(t)/\partial E_0$ greater or smaller for the wealthy? Exploring this question requires combining equations (83), (90), and (91) in Appendix A.5. There are many such relationships, given the many possible permutations. It is therefore impossible to discuss all of them, and their expressions can become quite involved. But the interested reader can use Appendix A.5 to delve further into specific relationships of interest.

The general lesson from this type of analysis is that variations in two parameters, Z_0 and W_0, of which at least one has a positive effect on longevity, often reinforce each other (complementarity), i.e. the total effect on a model outcome $g(t)$ is greater than the sum of the individual effects (see Appendix A.10 and Fonseca et al. 2013).

As a concrete example, consider the effect of life expectancy T on skill capital $E(t)$ (83 in Appendix A.5), an effect that has attracted much attention in both the theoretical and empirical literatures (e.g., Ben-Porath, 1967; Hazan, 2009; Jayachandran and Lleras-Muney, 2009; Fortson, 2011; Oster, Shoulson, and Dorsey, 2013). Expression (83) shows that the effect of life expectancy on skill capital $\partial E(t)/\partial T|_{Z_0}$ is reinforced by higher productivity of skill-capital investment $\mu_E(t)$ and by a higher relative marginal value of skill $q_{e/a}(t)$. The productivity factor $\mu_E(t)$ increases in the technology of skill investment production $\theta_E(t)$ and decreases in the price of skill-capital investment $p_E(t)$ (see 34). The value of skill $q_{e/a}(t)$ captures the future returns to skill investment (see 11). This depends on the resources at the individual's disposal, technology, institutions and markets. For example, if the demand for skilled labor is high (high wages $w[t, E(t)]$ for the skilled, existence of high-tech sectors, etc), then investing in skill has value.[32]

An implication is that one expects heterogeneity in the effect of longevity gains on skill and health, as a result of differences in institutions and environment. Responses to longevity gains are predicted to be small if the returns to education are small (e.g., in a society where an extractive and exclusive elite controls the nation's wealth; Deaton, 2013), and they would be large in societies where skill investment is productive and affordable, and the institutional environment is favorable. Indeed, Figure 2 shows that changes in life expectancy are negatively associated with changes in education in countries with severe impediments to development, while in the rest of the world the association is positive. Potentially, this heterogeneity may explain why some studies have found effects of longevity gains on skill formation (e.g., Bleakley, 2007; Jayachandran and Lleras-Muney,

e.g., institutions and income per capita, that could explain the association. See Appendix A.1 for details.

[32] In the simple model, the value of skill $q_{e/a}(t)$ increases in wealth A_0 (86), skill E_0 (91), health H_0 (97), skill production $\mu_E(t)$ (105), and health production $\mu_H(t)$ (114).

2009; Bleakley, 2010b; Fortson, 2011), while others have not (e.g., Acemoglu and Johnson, 2007; Cutler et al. 2010).

4 Discussion

This paper presents a theory of joint investment in skill capital, health capital, and longevity, with three distinct phases of life: schooling, work, and retirement. The theory brings together (or unifies) the skill- and health-capital literatures, encompassing canonical skill-capital theories such as those developed by Becker (1964) and Ben-Porath (1967), and canonical health-capital theories such as those developed by Grossman (1972a;1972b; 2000) and Ehrlich and Chuma (1990).

The contribution of this paper is twofold. First, by unifiying the skill- and health-capital literatures our theory provides new insight into the distinct characteristics of skill and health that have heretofore not been uncovered. Second, the theory provides a framework for explaining stylized facts and for deriving new predictions that can be explored in future research.

Human capital is multidimensional (Acemoglu and Autor, 2012), and skill and health are potentially its most important dimensions (Schultz, 1961; Grossman, 2000; Becker, 2007). Both skill and health are human-capital stocks that depreciate over time, and investing in them can (partially) counteract their deterioration. Skill and health share the defining characteristic of human capital that they make individuals more productive. Despite their similarities, there are some notable differences. Grossman (1972a; 1972b; 2000) has argued that health, in contrast to skill, provides a consumption benefit $\partial U/\partial H$ (direct utility) in addition to a production benefit $\partial Y/\partial H$. Ehrlich and Chuma (1990) have emphasized that health is also distinct from skill in that maintaining health extends life (see 14).

This paper suggests three important additional differences between skill and health. First, we argue that skill capital largely determines the rate of return per period (the wage rate), while health capital largely determines the period itself, determining the amount of time that can be devoted to work and other uses, not just within a day (as in Grossman, 1972a; 1972b) but over the entire lifecycle, determining the duration of the schooling, work, and retirement phases of life. Second, skill is valued early in life, while health is valued later in life (conjecture 2). An implication is that individuals will use the schooling period primarily to invest in skill, while retirement is mostly devoted to health investment and leisure. Third, skill formation is governed by a production process in which investment in skill increases the level of skill, at least initially, while health formation is governed by a production process where health eventually declines, no matter how much one invests in it (conjecture 1).

Recognizing that health is an essential, but distinct, component of human capital suggests misspecification in many empirical applications of human-capital theory. Examples include, but are not limited to, the importance of human capital in development-accounting efforts of economic growth, where the health component of human capital is typically ignored (e.g., Weil, 2007); and the attribution of the hump-shaped earnings profile over the lifecycle to skill-capital decline. Since earnings $Y(t)$ are the product of the wage rate $w[t, E(t)]$ and time spent working $\tau_w(t)$, earnings will decrease more rapidly than wages, as a result of declining health (increasing sick time $s[H(t))$. Thus, the observed hump-shaped earnings profile could be partially due to reduced time spent working as a result of declining health.[33]

Comparative dynamic analyses of the model show that (i) wealth, skill and health hardly affect human capital investment when additional resources cannot extend life (fixed T; generalized Heckman result), and (ii) that greater endowed wealth, skill, and health lead to higher investment, greater skill, better health, and a longer life (generalized Ehrlich and Chuma result).

We additionally obtain several novel predictions of the theory that we elaborated upon in the introduction and shortly summarize here. First, in contrast to the skill-capital and health-capital literatures, our model suggests that health affects skill formation, even after the schooling period. This is because health and skill are strongly complementary in generating earnings, healthy individuals are more efficient producers of skill, and since healthy individuals live longer their returns to skill investment are higher. These pathways are understudied in empirical as well as theoretical research (prediction 1). Second, our model predicts a central role for longevity. Additional resources (e.g., wealth, skill, health, permanent income) lead to more health and skill investment only if they are accompanied by an increase in longevity (prediction 2). Third, the model highlights that complementarity effects, operating through longevity, reinforce the associations between wealth, skill, health, and technology (prediction 3). That is, the combined effect is greater than the sum of the individual effects, potentially explaining the unprecedented increases in skill and health during the 20^{th} Century.

These are just a few examples of how the theory can be used as an analytical framework to study empirical research questions and to generate testable predictions. The theory is rich, and it is impossible to produce an exhaustive list of its possible uses. We hope the theory will aid researchers in studying their own particular questions of interest. For example, the analytical comparative dynamic expressions of the simpler model can be employed to not only study the sign of the comparative dynamic effects but also to provide information on its determinants (see Appendix section A.5). The discussion of prediction 3 provides an illustration of the potential of this type of analysis.

[33] Indeed Casanova (2013) finds that wages remain flat for two-thirds of workers till retirement, while the remaining one third has flat wages till they transition into partial or full retirement (often involuntarily, e.g., for health reasons).

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A Appendix

A.1 Data sources and analyses

Figure 1 The measures "Inequality in life expectancy" and "Inequality in education" are taken from the United Nations Human Development Report 2014, Table 3, p. 168-171 (http://hdr.undp.org/sites/default/files/hdr14-report-en-1.pdf). Population size for the weighted analyses are from the Barro Lee database (Barro and Lee, 2013; http://www.barrolee.com/). Both inequality measures are estimated using the Atkinson inequality index. In the computations the inequality aversion parameter is set to 1, such that the Atkinson Index A is defined as

$$A = 1 - \frac{g}{\mu}$$

where μ is the arithmetic mean and

$$g = \left(\prod_{i=1}^{N} y_i \right)^{1/N}$$

is the geometric mean of the variable of interest $y_i, i = 1, \ldots, N$. The range of the Atkinson index is from 0 to 1, with 0 being a fully equal distribution, and higher levels indicate more unequal distributions. For more information on the exact computation of the indices see technical note 2 of http://hdr.undp.org/sites/default/files/hdr14_technical_notes.pdf.

The relevant regression output accompanying Figure 1 is given in Table 2. The table

Table 2: Association between inequality in education and inequality in life expectancy

	Unweighted	Weighted
Inequality in life expectancy	0.856***	1.164***
	(0.051)	(0.231)
Intercept	5.097***	6.339**
	(1.125)	(3.016)
Number of observations	147	97

Dependent variable is Atkinson Inequality in Education. The results in the column 'weighted' are weighted by population size. Robust standard errors are in parentheses. * p-value < 0.1, ** p-value < 0.05, *** p-value < 0.01.

indicates that there is a strong positive association between inequality in life expectancy and inequality in education. Obviously, we cannot make any causal statements on the basis of this analysis, since causality could also run from inequality in education to inequality in life expectancy, and both inequality measures could be influenced by factors such as institutions, national income, etc. Nonetheless, the results in Table 2 and Figure 1 are consistent with the theoretical prediction that inequality in life expectancy results in inequality in human capital, here measured by education (though note that the other direction of causality and the potential role of third factors are also included in the theory).

Figure 2 The variable "Life Expectancy" is taken from the World Health Organization (WHO) Global Health Observatory Data (http://www.who.int/gho/mortality_burden_disease/life_tables/situation_trends/en/). We use life expectancy at birth in 1990 and in 2012 to compute changes in life expectancy. The variable "Education" is taken from the Barro Lee database (Barro and Lee, 2013; http://www.barrolee.com/). We use average years of education for individuals aged 15-24 in 1990 and 2010 to compute changes in education. Reassuringly, the results are very similar when using the entire age distribution instead of restricting the population to individuals aged 15-24 (results are available upon request).

The country grouping is done on the basis of the United Nations definition of *Least Developed Countries* (LDCs). LDCs are low-income countries confronting severe structural impediments to sustainable development, and are identified on the basis of gross national income per capita, the human asset index, and economic vulnerability. There are currently 48 countries designated by the United Nations as LDCs. See http://www.un.org/en/development/desa/policy/cdp/ldc_info.shtml for more information. Note that our results are not contingent upon the definition used – when using the World Bank definition of "Heavily Indebted Poor Countries" the results are very similar (results are available upon request).

The relevant regression output accompanying Figure 2 is given in Table 3. The point estimates indicate that the association between changes in life expectancy and changes in education is positive for developed countries, while negative for the least developed countries. Results are statistically significant when using weighted regressions by population size. Again, while the direction of causality cannot be inferred from this analysis, the result is consistent with the prediction that the effect of longevity on education depends on the productivity and affordability of educational investments, and the institutional environment, here proxied by the UN definition of least developed countries.

Table 3: Association between changes in education and changes in life expectancy

	Unweighted	Weighted
Changes in life expectancy	0.058	0.339***
	(0.057)	(0.121)
Changes in life expectancy*LDC	−0.125	−0.402**
	(0.091)	(0.160)
LDC	1.588	2.256
	(0.986)	(0.231)
Intercept	0.924***	0.091
	(0.301)	(3.016)
Number of observations	86	86

Dependent variable is Changes in Years of Education for the population aged 15-24 between 1990 and 2010. Changes in Life Expectancy at birth is measured between 1990 and 2012. LDC is a dummy variable indicating the Least Developed Countries as defined by the United Nations. The results in the column 'weighted' are weighted by population size. Robust standard errors are in parentheses. * p-value < 0.1, ** p-value < 0.05, *** p-value < 0.01.

A.2 First-order (necessary) conditions: general framework

The first-order necessary conditions for the optimal control problem, consisting of maximizing the objective function (1) subject to the constraints (2) to (4) and begin and end conditions, follow from Pontryagin's maximum principle (e.g., Caputo, 2005). The Hamiltonian is given by (10). For the co-state variable $q_A(t)$ associated with assets we have

$$\begin{aligned} \frac{\partial q_A}{\partial t} &= -\frac{\partial \Im}{\partial A} = -q_A(t) r \Leftrightarrow \\ q_A(t) &= q_A(0) e^{-rt}. \end{aligned} \tag{37}$$

The co-state variable $q_E(t)$ associated with skill capital follows from

$$\begin{aligned} \frac{\partial q_E}{\partial t} &= -\frac{\partial \Im}{\partial E} \\ &= -q_A(t)\frac{\partial Y}{\partial E} - q_H(t)\frac{\partial f_H}{\partial E} + q_E(t)\left[d_E(t) - \frac{\partial f_E}{\partial E}\right]. \quad S \leq t < R \end{aligned} \tag{38}$$

For non-working ages, income $Y(t)$ is fixed, and the evolution of the co-state variable $q_E(t)$ reduces to

$$\begin{aligned} \frac{\partial q_E}{\partial t} &= -\frac{\partial \Im}{\partial E} \\ &= -q_H(t)\frac{\partial f_H}{\partial E} + q_E(t)\left[d_E(t) - \frac{\partial f_E}{\partial E}\right]. \qquad 0 \leq t < S, R \leq t < T \end{aligned} \tag{39}$$

The co-state variable $q_H(t)$ associated with health capital follows from

$$\begin{aligned}
\frac{\partial q_H}{\partial t} &= -\frac{\partial \Im}{\partial H} \\
&= -\frac{\partial U}{\partial H} e^{-\beta t} - q_A(t) \frac{\partial Y}{\partial H} \\
&- q_E(t) \frac{\partial f_E}{\partial H} + q_H(t) \left[d_H(t) - \frac{\partial f_H}{\partial H} \right] - \lambda_{H_{\min}}(t). \qquad S \leq t < R
\end{aligned} \tag{40}$$

For non-working ages, income $Y(t)$ is fixed, and the evolution of the co-state variable $q_H(t)$ is given by

$$\begin{aligned}
\frac{\partial q_H}{\partial t} &= -\frac{\partial \Im}{\partial H} \\
&= -\frac{\partial U}{\partial H} e^{-\beta t} - q_E(t) \frac{\partial f_E}{\partial H} + q_H(t) \left[d_H(t) - \frac{\partial f_H}{\partial H} \right] \\
&+ \lambda_{\tau_w}(t) w[t, E(t)] \frac{\partial s}{\partial H} - \lambda_{H_{\min}}(t). \qquad 0 \leq t < S, R \leq t < T
\end{aligned} \tag{41}$$

where the cost of sick time is now valued at $\lambda_{\tau_w}(t)$.

The first-order condition for investment in skill capital (18) follows from optimizing with respect to skill-capital investment goods and services $X_E(t)$ and time inputs $\tau_E(t)$:

$$\frac{\partial \Im}{\partial X_E} = 0 \Leftrightarrow \quad q_E(t) \frac{\partial f_E}{\partial I_E} \frac{\partial I_E}{\partial X_E} - q_A(t) p_E(t) = 0 \tag{42}$$

$$\frac{\partial \Im}{\partial \tau_E} = 0 \Leftrightarrow \quad q_E(t) \frac{\partial f_E}{\partial I_E} \frac{\partial I_E}{\partial \tau_E} - q_A(t) w[t, E(t)] = 0 \qquad S \leq t < R \tag{43}$$

$$\Leftrightarrow \quad q_E(t) \frac{\partial f_E}{\partial I_E} \frac{\partial I_E}{\partial \tau_E} - \lambda_{\tau_w}(t) w[t, E(t)] = 0. \qquad 0 \leq t < S, R \leq t < T \tag{44}$$

The first-order condition for investment in health capital (23) follows from optimizing with respect to health investment goods and services $X_H(t)$ and time inputs $\tau_H(t)$:

$$\frac{\partial \Im}{\partial X_H} = 0 \Leftrightarrow \quad q_H(t) \frac{\partial f_H}{\partial I_H} \frac{\partial I_H}{\partial X_H} - q_A(t) p_H(t) = 0 \tag{45}$$

$$\frac{\partial \Im}{\partial \tau_H} = 0 \Leftrightarrow \quad q_H(t) \frac{\partial f_H}{\partial I_H} \frac{\partial I_H}{\partial \tau_H} - q_A(t) w[t, E(t)] = 0 \qquad S \leq t < R \tag{46}$$

$$\Leftrightarrow \quad q_H(t) \frac{\partial f_H}{\partial I_H} \frac{\partial I_H}{\partial \tau_H} - \lambda_{\tau_w}(t) w[t, E(t)] = 0. \qquad 0 \leq t < S, R \leq t < T \tag{47}$$

The first-order condition for consumption (15) follows from optimizing with respect to consumption goods and services $X_C(t)$:

$$\frac{\partial \Im}{\partial X_C} = 0 \Leftrightarrow \quad \frac{\partial U}{\partial X_C} e^{-\beta t} - q_A(t) p_C(t) = 0. \tag{48}$$

The first-order condition for leisure time (16) follows directly from optimizing with respect to leisure time $L(t)$:

$$\frac{\partial \Im}{\partial L} = 0 \Leftrightarrow \quad \frac{\partial U}{\partial L} e^{-\beta t} - q_A(t) w[t, E(t)] = 0 \qquad S \le t < R \tag{49}$$

$$\Leftrightarrow \quad \frac{\partial U}{\partial L} e^{-\beta t} - \lambda_{\tau_w}(t) w[t, E(t)] = 0. \qquad 0 \le t < S, R \le t < T \tag{50}$$

A.3 First-order (necessary) conditions: simpler model

The first-order conditions are obtained by taking the derivative of the Hamitonian

$$\Im = U\{X_C(t), L(t)[E(t) + H(t)]\} e^{-\beta t} + q_E(t)\frac{\partial E}{\partial t} + q_H(t)\frac{\partial H}{\partial t} + q_A(t)\frac{\partial A}{\partial t}, \tag{51}$$

with respect to the controls (not shown). Start with the first-order condition for the optimal expenditures on skill capital goods, $X_E(t)$, and for time inputs, $\tau_E(t)$, and divide the two resulting expressions by one another to obtain the relation

$$\tau_E(t)\,[E(t) + H(t)] = \frac{\alpha_E}{\beta_E} p_E(t) X_E(t). \tag{52}$$

Similarly for health investment one obtains the relation

$$\tau_H(t)\,[E(t) + H(t)] = \frac{\alpha_H}{\beta_H} p_H(t) X_H(t). \tag{53}$$

Now insert these relations back into the first-order condition for $X_E(t)$, $\tau_E(t)$, $X_H(t)$, and $\tau_H(t)$, to obtain the analytical solutions:

$$X_E(t) = \frac{\beta_E \mu_E(t)}{p_E(t)} q_{e/a}(t)^{\frac{1}{1-\gamma_E}}, \tag{54}$$

$$\tau_E(t)\,[E(t) + H(t)] = \alpha_E \mu_E(t) q_{e/a}(t)^{\frac{1}{1-\gamma_E}}, \tag{55}$$

$$X_H(t) = \frac{\beta_H \mu_H(t)}{p_H(t)} q_{h/a}(t)^{\frac{1}{1-\gamma_H}}, \tag{56}$$

$$\tau_H(t)\,[E(t) + H(t)] = \alpha_H \mu_H(t) q_{h/a}(t)^{\frac{1}{1-\gamma_H}}, \tag{57}$$

where $\gamma_E = \alpha_E + \beta_E$, and $\gamma_H = \alpha_H + \beta_H$, and the functions $\mu_E(t)$ and $\mu_H(t)$ are defined in (34) and (35).

The co-state equations for $q_E(t)$ and $q_H(t)$ follow from the usual conditions $\partial q_E/\partial t = -\partial \Im/\partial E$ and $\partial q_H/\partial t = -\partial \Im/\partial H$, and using (54) to (57), we obtain

$$\frac{\partial q_E}{\partial t} = q_E(t) d_E(t) - q_A(t), \tag{58}$$

$$\frac{\partial q_H}{\partial t} = q_H(t) d_H(t) - q_A(t). \tag{59}$$

The convenient choices made for the functional forms, referred to as "Ben-Porath neutrality" ensure that the relative marginal value of skill capital $q_{e/a}(t)$, and in our case also of health capital $q_{h/a}(t)$, are independent of the capital stocks (see 58 and 59). The system of equations for (the relative marginal value of) skill capital, and (the relative marginal value of) health capital reduces to the following system:

$$\frac{\partial q_{e/a}}{\partial t} = q_{e/a}(t)\left[d_E(t)+r\right]-1, \tag{60}$$

$$\frac{\partial E}{\partial t} = \mu_E(t)q_{e/a}(t)^{\frac{\gamma_E}{1-\gamma_E}}-d_E(t)E(t), \tag{61}$$

$$\frac{\partial q_{h/a}}{\partial t} = q_{h/a}(t)\left[d_H(t)+r\right]-1, \tag{62}$$

$$\frac{\partial H}{\partial t} = \mu_H(t)q_{h/a}(t)^{\frac{\gamma_H}{1-\gamma_H}}-d_H(t)H(t). \tag{63}$$

Using the dynamic relation for skill- (2) and health-capital formation (3), the Ben-Porath production functions (32) and (33), and the solutions for the controls (54) to (57), one obtains analytical expressions for the relative marginal value of skill capital $q_{e/a}(t)$, skill capital $E(t)$, the relative marginal value of health capital $q_{h/a}(t)$, and health capital $H(t)$:

$$q_{e/a}(t) = \int_t^T e^{-\int_t^s [d_E(x)+r]dx}ds, \tag{64}$$

$$E(t) = E_0 e^{-\int_0^t d_E(x)dx}+\int_0^t \mu_E(s)q_{e/a}(s)^{\frac{\gamma_E}{1-\gamma_E}}e^{-\int_s^t d_E(x)dx}ds, \tag{65}$$

$$q_{h/a}(t) = q_{h/a}(0)e^{\int_0^t [d_H(x)+r]dx}-\int_0^t e^{\int_s^t [d_H(x)+r]dx}ds, \tag{66}$$

$$H(t) = H_0 e^{-\int_0^t d_H(x)dx}+\int_0^t \mu_H(s)q_{h/a}(s)^{\frac{\gamma_H}{1-\gamma_H}}e^{-\int_s^t d_H(x)dx}ds, \tag{67}$$

where we have used $q_{e/a}(T) = 0$, and the solution for the marginal value of assets $q_A(t) = q_A(0)e^{-rt}$, see (37).

Using the dynamic relation for assets (4), (31), and (54) to (71), we obtain

$$\begin{aligned} A(t)e^{-rt} &= A_0+\int_0^t e^{-rs}\left[E(s)+H(s)\right]ds \\ &- \gamma_E\int_0^t \mu_E(s)q_{e/a}(s)^{\frac{1}{1-\gamma_E}}e^{-rs}ds \\ &- \gamma_H\int_0^t \mu_H(s)q_{h/a}(s)^{\frac{1}{1-\gamma_H}}e^{-rs}ds \\ &- q_A(0)^{-1/\rho}\Lambda\int_0^t p_C(s)^{-\zeta(1-\rho)/\rho}e^{-\frac{(\beta-r(1-\rho)r)}{\rho}s}ds. \end{aligned} \tag{68}$$

Finally, the analytical solutions for consumption $X_C(t)$ and leisure $L(t)\,[E(t)+H(t)]$ are obtained by dividing the two first-order conditions, leading to the relation

$$L(t)\,[E(t)+H(t)] = \frac{(1-\zeta)}{\zeta} p_C(t) X_C(t). \quad (69)$$

Inserting this relation back into the first-order conditions for consumption $X_C(t)$ and leisure $L(t)$, leads to the analytical solutions

$$X_C(t) = \zeta \Lambda q_A(0)^{-1/\rho} p_C(t)^{-(1-\zeta+\zeta/\rho)} e^{-\frac{\beta-r}{\rho}t}, \quad (70)$$

$$L(t)[E(t)+H(t)] = (1-\zeta)\Lambda q_A(0)^{-1/\rho} p_C(t)^{-\zeta(1-\rho)/\rho} e^{-\frac{\beta-r}{\rho}t}, \quad (71)$$

where

$$\Lambda \equiv \left[\zeta^{\zeta}(1-\zeta)^{1-\zeta}\right]^{\frac{1-\rho}{\rho}}. \quad (72)$$

The analytical solutions for the controls, state variables, and co-state variables (54) to (71), are functions of the marginal value of initial wealth $q_A(0)$, the initial relative marginal value of skill-capital $q_{e/a}(0)$, and the initial relative marginal value of health-capital $q_{h/a}(0)$. These in turn are determined by initial, end, and transversality conditions.

From (68), and the initial, $A(0) = A_0$, and end condition, $A(T) = A_T$, follows a condition for $q_A(0)$

$$\begin{aligned} A_T e^{-rT} &= A_0 + \int_0^T e^{-rs}\,[E(s)+H(s)]\,ds \\ &- \gamma_E \int_0^T \mu_E(s) q_{e/a}(s)^{\frac{1}{1-\gamma_E}} e^{-rs} ds \\ &- \gamma_H \int_0^T \mu_H(s) q_{h/a}(s)^{\frac{1}{1-\gamma_H}} e^{-rs} ds \\ &- q_A(0)^{-1/\rho}\Lambda \int_0^T p_C(s)^{-\zeta(1-\rho)/\rho} e^{-\frac{(\beta-r(1-\rho))}{\rho}s} ds. \end{aligned} \quad (73)$$

From (67) and the initial, $H(0) = H_0$, and end condition, $H(T) = H_{\min}$, follows a condition for $q_{h/a}(0)$

$$H_{\min} e^{\int_0^T d_H(x)dx} = H_0 + \int_0^T \mu_H(s) q_{h/a}(s)^{\frac{\gamma_H}{1-\gamma_H}} e^{\int_0^s d_H(x)dx} ds. \quad (74)$$

The condition for $q_{e/a}(0)$ follows from the transversality condition $q_E(T) = 0$ ($E(T)$ free) and is obtained from (64)

$$q_{e/a}(0) = \int_0^T e^{-\int_0^s [d_E(x)+r]dx} ds. \quad (75)$$

The remaining endogenous parameters and functions in the above three conditions (73), (74), and (75), are T, which is determined by (14), $q_{e/a}(t)$, which is determined by (64), $E(t)$, which is determined by (65), $q_{h/a}(t)$, which is determined by (66), and $H(t)$, which is determined by (67).

A.4 Comparison with the general theory

The simpler version of our model maintains the most important properties of the general model defined in section 2.1. There are some costs associated with the simplications associated with the assumption of "Ben-Porath neutrality" (see below), which we describe here, but the benefits arguably outweigh the costs. Most importantly, the assumption enables obtaining analytical results for the comparative dynamic analyses.

As for the general model, in the simpler model both skill and health contribute to earnings, the production processes of skill and health investment are increasing and concave in the investment inputs,[34] and they exhibit both self-reinforcing and cross-fertilizing self-productivity and dynamic complementarity. Not surprisingly, the dynamics of the simpler model are qualitatively similar to that of the general model. The relative marginal value of skill decreases with age (investment decreases with shortening of the horizon) and skill capital increases. The relative marginal value of health increases with age (investment in health increases with age till death) and health capital declines (conjectures 1 and 2).

Compared to the general theory, there are however a few differences. First, the assumed specific functional form for the utility, earnings, skill-production, and health-production functions, ensure that the marginal value of skill and of health are no longer functions of the stock of skill and health (compare 20 and 25 with 60 and 62). This is commonly known as "Ben-Porath neutrality", and as a result we can solve the model analytically.[35] In the general model, however, the relative marginal value of skill is likely to be decreasing in the stock of skill (due to decreasing returns to scale) and potentially increasing in the stock of health (due to complementarity between skill and health, in the generation of earnings, and in the production of skill and health investment $f_E(t)$ and $f_H(t)$, see 20). The opposite is true for the relative marginal value of health (see 25), which is likely decreasing in health and potentially increasing in skill.

Second, we assume that there are no separate periods exclusively devoted to schooling S and to retirement R. While the model no longer contains an explicit school-leaving age and retirement age, schooling and retirement phases do exist. Early in life individuals invest in skill capital as the stock of skill is low (and hence the marginal benefits high), the opportunity cost of time is low, and the horizon over which the benefits of skill-capital investment can be reaped is long. Individuals do not work much as low skill capital implies low earnings, such that this period of life corresponds to a schooling phase. As individuals develop skill capital they start investing less in skill due to gradually declining marginal benefits and shortening of the horizon, and work more (working phase). Later in life individuals work less and invest more in health as a result of declining health,

[34] For $\alpha_E + \beta_E < 1$ and $\alpha_H + \beta_H < 1$, we have $\partial f_E/\partial X_E > 0$, $\partial f_E/\partial \tau_E > 0$, $\partial f_H/\partial X_H > 0$, $\partial f_H/\partial \tau_H > 0$, $\partial^2 f_E/\partial X_E^2 < 0$, $\partial^2 f_E/\partial \tau_E^2 < 0$, $\partial^2 f_H/\partial X_H^2 < 0$, $\partial^2 f_H/\partial \tau_H^2 < 0$, $\left(\partial^2 f_E/\partial X_E^2\right)\left(\partial^2 f_E/\partial \tau_E^2\right) > \left(\partial^2 f_E/\partial X_E \partial \tau_E\right)^2$ and $\left(\partial^2 f_H/\partial X_H^2\right)\left(\partial^2 f_H/\partial \tau_H^2\right) > \left(\partial^2 f_H/\partial X_H \partial \tau_H\right)^2$.

[35] To maintain Ben-Porath neutrality, skill also enters the utility function in the simpler version of the model, so that not only health, but also skill provides a consumption benefit.

corresponding to a retirement phase. Thus the simpler model contains phases of schooling, work and retirement. The institutions of schooling and retirement, defined in the general model, only formalize and exacerbate this natural pattern.

Third, the simpler model assumes no sick time. Therefore health $H(t)$ does not protect time per period (let's say during a day), and the simpler model loses the characteristic of earnings being multiplicative in skill and health. Both health and skill still contribute to earnings, but do so in an additive way.[36] Since we find strong complementarity effects between skill and health even for the simpler model, the general model would only excacerbate these, but would not lead to a different conclusion.

A.5 Comparative dynamics: simpler model

Consider a generic control, state, or co-state function $g(t)$ and a generic variation δZ_0 in an initial condition or model parameter. The effect of the variation δZ_0 on the optimal path of $g(t)$ can be broken down into variation for fixed longevity T and variation due to the resulting change in the horizon T (see 36). In the below analyses (i) we first analyze the case for fixed T, from which we obtain $\partial g(t)/\partial Z_0|_T$ (see discussion below), (ii) we then determine $\partial T/\partial Z_0$, and (iii) last we obtain $\partial g(t)/\partial T|_{Z_0}$, so that we obtain the full comparative dynamic effect.

Comparative dynamics of length of life $\partial T/\partial Z_0$: For fixed length of life T we can take derivatives of the first-order conditions and state equations with respect to the initial condition or model parameter and study the optimal adjustment to the lifecycle path in response to variation in an initial endowment or other model parameter.

For free T, however, this is slightly more involved since the additional condition $\Im(T) = 0$ has to be satisfied. Varying the initial condition or model parameter Z_0, and taking into account $\Im(T) = 0$, we have

$$\left.\frac{\partial \Im(T)}{\partial Z_0}\right|_T \delta Z_0 + \left.\frac{\partial \Im(T)}{\partial T}\right|_{Z_0} \delta T = 0. \tag{76}$$

Using the expression for the Hamiltonian (51), taking the first derivative of the transversality condition $\Im(T) = 0$ with respect to the initial conditions or model parameter Z_0, and holding length of life T fixed, we obtain

[36] For simplicity we assume constant returns to scale of human capital $E(t) + H(t)$ in the production of wages $w[t, E(t), H(t)]$. Predictions are however not affected when imposing decreasing or increasing returns to scale $w[t, E(t), H(t)] = [E(t) + H(t)]^\sigma$ with $\sigma \neq 1$, as long as human capital affects the utility of leisure $U\{X_C(t), L(t)\,[E(t) + H(t)]^\sigma\}$, and the efficiency of time investments $\tau_E(t)\,[E(t) + H(t)]^\sigma$ and $\tau_H(t)\,[E(t) + H(t)]^\sigma$ in the production functions of skill capital and health capital in the same way.

$$
\begin{aligned}
\left.\frac{\partial \Im(T)}{\partial Z_0}\right|_T &= \frac{\partial \Im}{\partial \xi}\left.\frac{\partial \xi(T)}{\partial Z_0}\right|_T + \frac{\partial \Im}{\partial E}\left.\frac{\partial E(T)}{\partial Z_0}\right|_T + \frac{\partial \Im}{\partial A}\left.\frac{\partial A(T)}{\partial Z_0}\right|_T + \frac{\partial \Im}{\partial H}\left.\frac{\partial H(T)}{\partial Z_0}\right|_T \\
&\quad + \frac{\partial \Im}{\partial q_E}\left.\frac{\partial q_E(T)}{\partial Z_0}\right|_T + \frac{\partial \Im}{\partial q_A}\left.\frac{\partial q_A(T)}{\partial Z_0}\right|_T + \frac{\partial \Im}{\partial q_H}\left.\frac{\partial q_H(T)}{\partial Z_0}\right|_T \\
&= -\left.\frac{\partial q_E(t)}{\partial t}\right|_{T,t=T}\left.\frac{\partial E(T)}{\partial Z_0}\right|_T + \left.\frac{\partial q_A(T)}{\partial Z_0}\right|_T\left.\frac{\partial A(t)}{\partial t}\right|_{T,t=T} \\
&\quad + \left.\frac{\partial q_H(T)}{\partial Z_0}\right|_T\left.\frac{\partial H(t)}{\partial t}\right|_{T,t=T}, \qquad (77)
\end{aligned}
$$

where $\xi(t)$ is the vector of control functions $X_C(t)$, $L(t)$, $X_E(t)$, $\tau_E(t)$, $X_H(t)$, and $\tau_H(t)$. The first-order conditions imply $\partial\Im(t)/\partial\xi(t) = 0$. Further, $\partial\Im(T)/\partial E = -\left.\partial q_E(t)/\partial t\right|_{t=T}$, $\partial A(T)/\partial Z_0 = \partial H(T)/\partial Z_0 = 0$ since $A(T)$ and $H(T)$ are fixed, and $\left.\partial q_E(T)/\partial Z_0\right|_T = 0$ since $q_E(T) = 0$.

Note that we distinguish in notation between $\left.\partial f(t)/\partial t\right|_{t=T}$, which represents the derivative with respect to time t at time $t = T$, and $\left.\partial f(t)/\partial T\right|_{t=T}$, which represents variation with respect to the parameter T at time $t = T$.

From (76) and (77) we have

$$
\frac{\partial T}{\partial Z_0} = \frac{q_A(T)\left.\frac{\partial E(T)}{\partial Z_0}\right|_T + \left.\frac{\partial q_A(T)}{\partial Z_0}\right|_T\left.\frac{\partial A(t)}{\partial t}\right|_{T,t=T} + \left.\frac{\partial q_H(T)}{\partial Z_0}\right|_T\left.\frac{\partial H(t)}{\partial t}\right|_{T,t=T}}{-\left.\frac{\partial \Im(T)}{\partial T}\right|_{Z_0}}, \qquad (78)
$$

where we have used $\left.\partial q_E(t)/\partial t\right|_{t=T} = q_E(T)d_E(T) - q_A(T) = -q_A(T)$ (see 58 and use $q_E(T) = 0$).

The denominator of (78) can be obtained from

$$
\begin{aligned}
\left.\frac{\partial \Im(T)}{\partial T}\right|_{Z_0} &= -\beta U[.]e^{-\beta T} \\
&\quad + q_A(T)\left.\frac{\partial E(T)}{\partial T}\right|_{Z_0} + \left.\frac{\partial q_A(T)}{\partial T}\right|_{Z_0}\left.\frac{\partial A(t)}{\partial t}\right|_{t=T} + \left.\frac{\partial q_H(T)}{\partial T}\right|_{Z_0}\left.\frac{\partial H(t)}{\partial t}\right|_{t=T}, \qquad (79)
\end{aligned}
$$

which follows from differentiating (51) with respect to T and using the first-order conditions (54) to (71), the co-state equations (60) to (63), (37), and the transversality condition $q_{e/a}(T) = 0$.

Consistent with diminishing returns to life extension (Ehrlich and Chuma, 1990), we assume

$$
\left.\frac{\partial \Im(T)}{\partial T}\right|_{Z_0} < 0, \qquad (80)
$$

in which case we can identify the sign of the variation in life expectancy from

$$\text{sign}\left(\frac{\partial T}{\partial Z_0}\right) = \text{sign}\left(\left.\frac{\partial \Im(T)}{\partial Z_0}\right|_T\right), \tag{81}$$

where,

$$\left.\frac{\partial \Im(T)}{\partial Z_0}\right|_T = q_A(T)\left.\frac{\partial E(T)}{\partial Z_0}\right|_T + \left.\frac{\partial q_A(T)}{\partial Z_0}\right|_T \left.\frac{\partial A(t)}{\partial t}\right|_{T,t=T} + \left.\frac{\partial q_H(T)}{\partial Z_0}\right|_T \left.\frac{\partial H(t)}{\partial t}\right|_{T,t=T}. \tag{82}$$

As (81) shows, we can explore variation in initial conditions keeping length of life T initially fixed in order to investigate whether life would be extended as a result of such variation.

Comparative dynamics of variation in length of life $\partial g(t)/\partial T|_{Z_0}$: The derivatives of the control functions, state function and co-state functions with respect to length of life T, holding constant Z_0, are identical for any initial condition or model parameter Z_0. We therefore first obtain their derivatives (using 64 to 75). The symbol $\gtreqless 0$ is used to indicate that the sign cannot unambiguously be determined.

$$\begin{aligned}
\left.\frac{\partial q_{e/a}(t)}{\partial T}\right|_{Z_0} &= e^{-\int_t^T [d_E(x)+r]dx} > 0, \\
\left.\frac{\partial q_{h/a}(t)}{\partial T}\right|_{Z_0} &= \frac{-\left.\frac{\partial H(t)}{\partial t}\right|_{t=T} e^{\int_0^t [2d_H(x)+r]dx} e^{\int_t^T d_H(x)dx}}{\frac{\gamma_H}{1-\gamma_H}\int_0^T \mu_H(s) q_{h/a}(s)^{\frac{2\gamma_H-1}{1-\gamma_H}} e^{\int_0^s [2d_H(x)+r]dx} ds} > 0, \\
\left.\frac{\partial E(t)}{\partial T}\right|_{Z_0} &= \frac{\gamma_E}{1-\gamma_E}\int_0^t \mu_E(s) q_{e/a}(s)^{\frac{2\gamma_E-1}{1-\gamma_E}} \left.\frac{\partial q_{e/a}(s)}{\partial T}\right|_{Z_0} e^{-\int_s^t d_E(x)dx} ds > 0, \\
\left.\frac{\partial H(t)}{\partial T}\right|_{Z_0} &= \frac{\gamma_H}{1-\gamma_H}\int_0^t \mu_H(s) q_{h/a}(s)^{\frac{2\gamma_H-1}{1-\gamma_H}} \left.\frac{\partial q_{h/a}(s)}{\partial T}\right|_{Z_0} e^{-\int_s^t d_H(x)dx} ds > 0, \\
\left.\frac{\partial X_E(t)}{\partial T}\right|_{Z_0} &= \frac{1}{1-\gamma_E}\frac{\beta_E}{p_E(t)}\mu_E(t) q_{e/a}(t)^{\frac{\gamma_E}{1-\gamma_E}} \left.\frac{\partial q_{e/a}(t)}{\partial T}\right|_{Z_0} > 0, \\
\left.\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{Z_0} &= \frac{1}{1-\gamma_E}\alpha_E \mu_E(t) q_{e/a}(t)^{\frac{\gamma_E}{1-\gamma_E}} \left.\frac{\partial q_{e/a}(t)}{\partial T}\right|_{Z_0} > 0, \\
\left.\frac{\partial X_H(t)}{\partial T}\right|_{Z_0} &= \frac{1}{1-\gamma_H}\frac{\beta_H}{p_H(t)}\mu_H(t) q_{h/a}(t)^{\frac{\gamma_H}{1-\gamma_H}} \left.\frac{\partial q_{h/a}(t)}{\partial T}\right|_{Z_0} > 0, \\
\left.\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{Z_0} &= \frac{1}{1-\gamma_H}\alpha_H \mu_H(t) q_{h/a}(t)^{\frac{\gamma_H}{1-\gamma_H}} \left.\frac{\partial q_{h/a}(t)}{\partial T}\right|_{Z_0} > 0,
\end{aligned}$$

$$\left.\frac{\partial q_A(0)}{\partial T}\right|_{Z_0} = \frac{-\left.\frac{\partial A(t)}{\partial t}\right|_{Z_0,t=T} e^{-rT} - \int_0^T \left.\frac{\partial \phi(s)}{\partial T}\right|_{Z_0} ds}{\frac{\Lambda}{\rho} q_A(0)^{-\left(\frac{1+\rho}{\rho}\right)} \int_0^T p_C(s)^{-\frac{\zeta(1-\rho)}{\rho}} e^{-\frac{(\beta - r(1-\rho))}{\rho}s} ds} \gtreqless 0,$$

$$\left.\frac{\partial A(t)}{\partial T}\right|_{Z_0} = e^{rt} \int_0^t \left.\frac{\partial \phi(s)}{\partial T}\right|_{Z_0} + \left[e^{rt} \frac{\Lambda}{\rho} q_A(0)^{-\frac{(1+\rho)}{\rho}} \int_0^t p_C(s)^{-\frac{\zeta(1-\rho)}{\rho}} e^{-\frac{(\beta - r(1-\rho))}{\rho}s} ds\right] \left.\frac{\partial q_A(0)}{\partial T}\right|_{Z_0} \gtreqless 0,$$

where

$$\left.\frac{\partial \phi(s)}{\partial T}\right|_{Z_0} \equiv e^{-rs}\left[\left.\frac{\partial E(s)}{\partial T}\right|_{Z_0} + \left.\frac{\partial H(s)}{\partial T}\right|_{Z_0}\right] - \frac{\gamma_E}{1-\gamma_E} e^{-rs} \mu_E(s) q_{e/a}(s)^{\frac{\gamma_E}{1-\gamma_E}} \left.\frac{\partial q_{e/a}(s)}{\partial T}\right|_{Z_0} - \frac{\gamma_H}{1-\gamma_H} e^{-rs} \mu_H(s) q_{h/a}(s)^{\frac{\gamma_H}{1-\gamma_H}} \left.\frac{\partial q_{h/a}(s)}{\partial T}\right|_{Z_0},$$

$$\left.\frac{\partial X_C(t)}{\partial T}\right|_{Z_0} = -\frac{\zeta\Lambda}{\rho} q_A(0)^{-\left(\frac{1+\rho}{\rho}\right)} p_C(t)^{-\left(\frac{\rho+\zeta-\zeta\rho}{\rho}\right)} e^{-\left(\frac{\beta-r}{\rho}\right)t} \left.\frac{\partial q_A(0)}{\partial T}\right|_{Z_0} \gtreqless 0,$$

$$\left.\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{Z_0} = -\frac{(1-\zeta)\Lambda}{\rho} q_A(0)^{-\left(\frac{1+\rho}{\rho}\right)} p_C(t)^{-\left(\frac{\zeta-\zeta\rho}{\rho}\right)} e^{-\left(\frac{\beta-r}{\rho}\right)t} \left.\frac{\partial q_A(0)}{\partial T}\right|_{Z_0} \gtreqless 0. \quad (83)$$

Comparative dynamics of initial wealth $\partial g(t)/\partial A_0$: First consider the case where T is fixed. Differentiating (74) with respect to A_0, using (66), and differentiating (75) with respect to A_0, one finds $\partial q_{e/a}(0)/\partial A_0\big|_T = 0$ and $\partial q_{h/a}(0)/\partial A_0\big|_T = 0$. Using (54) to (68), and (73), we obtain

$$\left.\frac{\partial q_{e/a}(t)}{\partial A_0}\right|_T = 0, \ \forall t \qquad \left.\frac{\partial q_{h/a}(t)}{\partial A_0}\right|_T = 0, \ \forall t$$

$$\left.\frac{\partial E(t)}{\partial A_0}\right|_T = 0, \ \forall t \qquad \left.\frac{\partial H(t)}{\partial A_0}\right|_T = 0, \ \forall t$$

$$\left.\frac{\partial X_E(t)}{\partial A_0}\right|_T = 0, \ \forall t \qquad \left.\frac{\partial X_H(t)}{\partial A_0}\right|_T = 0, \ \forall t$$

$$\left.\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial A_0}\right|_T = 0, \ \forall t \qquad \left.\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial A_0}\right|_T = 0, \ \forall t$$

$$
\begin{aligned}
\left.\frac{\partial A(t)}{\partial A_0}\right|_T &= e^{rt}\left[1-\frac{\int_0^t p_C(s)^{-\zeta(1-\rho)/\rho}e^{-\frac{(\beta-r(1-\rho))}{\rho}s}ds}{\int_0^T p_C(s)^{-\zeta(1-\rho)/\rho}e^{-\frac{(\beta-r(1-\rho))}{\rho}s}ds}\right]\geq 0,\\
\left.\frac{\partial q_A(0)}{\partial A_0}\right|_T &= \frac{-1}{\frac{\Lambda q_A(0)^{-\frac{(1+\rho)}{\rho}}}{\rho}\int_0^T p_C(s)^{-\frac{\zeta(1-\rho)}{\rho}}e^{-\frac{(\beta-r(1-\rho))}{\rho}s}ds}<0,\\
\left.\frac{\partial X_C(t)}{\partial A_0}\right|_T &= -\frac{1}{\rho}\zeta\Lambda q_A(0)^{-\frac{(1+\rho)}{\rho}}p_C(t)^{-(1-\zeta+\zeta/\rho)}e^{-\frac{\beta-r}{\rho}t}\left.\frac{\partial q_A(0)}{\partial A_0}\right|_T\\
&= \frac{\zeta p_C(t)^{-(1-\zeta+\zeta/\rho)}e^{-\frac{\beta-r}{\rho}t}}{\int_0^T p_C(s)^{-(1-\zeta+\zeta/\rho)}e^{-\frac{(\beta-r(1-\rho))}{\rho}s}ds}>0,\\
\left.\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial A_0}\right|_T &= -\frac{1}{\rho}\left(1-\zeta\right)\Lambda q_A(0)^{-\frac{(1+\rho)}{\rho}}p_C(t)^{-\zeta(1-\rho)/\rho}e^{-\frac{\beta-r}{\rho}t}\left.\frac{\partial q_A(0)}{\partial A_0}\right|_T\\
&= \frac{\left(1-\zeta\right)p_C(t)^{-\zeta(1-\rho)/\rho}e^{-\frac{\beta-r}{\rho}t}}{\int_0^T p_C(s)^{-\zeta(1-\rho)/\rho}e^{-\frac{(\beta-r(1-\rho))}{\rho}s}ds}>0. \qquad (84)
\end{aligned}
$$

Note that the relation for the variation in wealth has the desired properties $\partial A(0)/\partial A_0|_T = 1$, and $\partial A(T)/\partial A_0|_T = 0$.

Now allow length of life T to be optimally chosen. Using (78) we have

$$
\frac{\partial T}{\partial A_0} = \frac{\left.\frac{\partial q_A(0)}{\partial A_0}\right|_T e^{-rT}\left[\left.\frac{\partial A(t)}{\partial t}\right|_{t=T}+q_{h/a}(T)\left.\frac{\partial H(t)}{\partial t}\right|_{t=T}\right]}{-\partial\Im(T)/\partial T|_{A_0}}>0, \qquad (85)
$$

where we have used $\partial E(T)/\partial A_0|_T = 0$ (see 65 and note that $\partial q_{e/a}(t)/\partial A_0\big|_T = 0,\ \forall t$), $\partial q_H(T)/\partial A_0|_T = q_{h/a}(T)\ \partial q_A(T)/\partial A_0|_T$ (since $\partial q_{h/a}(T)/\partial A_0\big|_T = 0$), $\partial H(t)/\partial t|_{t=T} < 0$ by definition as health approaches $H_{\min}$ from above, $\partial A(t)/\partial t|_{t=T} < 0$ as individuals draw from their savings in old age, and $-\partial\Im(T)/\partial T|_{A_0} > 0$ (see 80).

Using (36), we obtain the following total responses to variation in wealth

$$
\begin{aligned}
\frac{\partial q_{e/a}(t)}{\partial A_0} &= \left.\frac{\partial q_{e/a}(t)}{\partial T}\right|_{A_0}\frac{\partial T}{\partial A_0}>0, &\qquad \frac{\partial q_{h/a}(t)}{\partial A_0} &= \left.\frac{\partial q_{h/a}(t)}{\partial T}\right|_{A_0}\frac{\partial T}{\partial A_0}>0,\\
\frac{\partial E(t)}{\partial A_0} &= \left.\frac{\partial E(t)}{\partial T}\right|_{A_0}\frac{\partial T}{\partial A_0}>0, &\qquad \frac{\partial H(t)}{\partial A_0} &= \left.\frac{\partial H(t)}{\partial T}\right|_{A_0}\frac{\partial T}{\partial A_0}>0,\\
\frac{\partial X_E(t)}{\partial A_0} &= \left.\frac{\partial X_E(t)}{\partial T}\right|_{A_0}\frac{\partial T}{\partial A_0}>0, &\qquad \frac{\partial X_H(t)}{\partial A_0} &= \left.\frac{\partial X_H(t)}{\partial T}\right|_{A_0}\frac{\partial T}{\partial A_0}>0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\tau_E(t)\left[E(t)+H(t)\right]}{\partial A_0} &= \left.\frac{\partial\tau_E(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{A_0}\frac{\partial T}{\partial A_0}>0,\\
\frac{\partial\tau_H(t)\left[E(t)+H(t)\right]}{\partial A_0} &= \left.\frac{\partial\tau_H(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{A_0}\frac{\partial T}{\partial A_0}>0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial A(t)}{\partial A_0} &= \left.\frac{\partial A(t)}{\partial A_0}\right|_T + \left.\frac{\partial A(t)}{\partial T}\right|_{A_0}\frac{\partial T}{\partial A_0} \gtreqless 0,\\
\frac{\partial q_A(0)}{\partial A_0} &= \left.\frac{\partial q_A(0)}{\partial A_0}\right|_T + \left.\frac{\partial q_A(0)}{\partial T}\right|_{A_0}\frac{\partial T}{\partial A_0} \gtreqless 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial X_C(t)}{\partial A_0} &= -\left(\frac{1}{\rho}\zeta\Lambda q_A(0)^{-\frac{(1+\rho)}{\rho}} p_C(t)^{-(1-\zeta+\zeta/\rho)} e^{-\frac{\beta-r}{\rho}t}\right)\times\\
&\qquad \left[\left.\frac{\partial q_A(0)}{\partial A_0}\right|_T + \left.\frac{\partial q_A(0)}{\partial T}\right|_{A_0}\frac{\partial T}{\partial A_0}\right] \gtreqless 0,\\
\frac{\partial L(t)\,[E(t)+H(t)]}{\partial A_0} &= -\left(\frac{1}{\rho}\left(1-\zeta\right)\Lambda q_A(0)^{-\frac{(1+\rho)}{\rho}} p_C(t)^{-\zeta(1-\rho)/\rho} e^{-\frac{\beta-r}{\rho}t}\right)\times\\
&\qquad \left[\left.\frac{\partial q_A(0)}{\partial A_0}\right|_T + \left.\frac{\partial q_A(0)}{\partial T}\right|_{A_0}\frac{\partial T}{\partial A_0}\right] \gtreqless 0, \qquad (86)
\end{aligned}
$$

where we have used (83). Note that the total response of $q_A(0)$ with respect to initial wealth A_0 is ambiguous, since the additional wealth has to be spread over more time periods ($\partial T/\partial A_0 > 0$). But, a longer horizon also increases the returns to skill investment and to health investment, increasing the stocks, earnings and permanent income (lowering the marginal value of wealth $q_A(0)$). Hence, the effect of initial wealth on $q_A(0)$ and thereby on consumption and leisure is ambiguous for free T. Since wealthy individuals are generally found to consume more and retire earlier (e.g., Imbens, Rubin and Sacerdote, 2001; Juster et al. 2006; Brown, Coile, and Weisbenner, 2010), it is plausible that the wealth effect dominates $\partial q_A(0)/\partial A_0 < 0$, and consumption goods and services $X_C(t)$ and effective leisure $L(t)\,[E(t)+H(t)]$ are higher throughout life.

Comparative dynamics of initial skill $\partial g(t)/\partial E_0$: Again, first consider the case where T is fixed. Differentiating (75) with respect to E_0, one finds $\partial q_{e/a}(0)/\partial E_0\big|_T = 0$ and differentiating (74) with respect to E_0, using (66) we find $\partial q_{h/a}(0)/\partial E_0\big|_T = 0$. Using (54) to (68), and (73), we find

$$
\begin{aligned}
\left.\frac{\partial q_{e/a}(t)}{\partial E_0}\right|_T &= 0,\ \forall t & \left.\frac{\partial q_{h/a}(t)}{\partial E_0}\right|_T &= 0,\ \forall t\\
\left.\frac{\partial E(t)}{\partial E_0}\right|_T &= e^{-\int_0^t d_E(x)dx} > 0,\ \forall t & \left.\frac{\partial H(t)}{\partial E_0}\right|_T &= 0,\ \forall t\\
\left.\frac{\partial X_E(t)}{\partial E_0}\right|_T &= 0,\ \forall t & \left.\frac{\partial X_H(t)}{\partial E_0}\right|_T &= 0,\ \forall t\\
\left.\frac{\partial \tau_E(t)\,[E(t)+H(t)]}{\partial E_0}\right|_T &= 0,\ \forall t & \left.\frac{\partial \tau_H(t)\,[E(t)+H(t)]}{\partial E_0}\right|_T &= 0,\ \forall t
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial A(t)}{\partial E_0}\right|_T &= e^{rt}\left[\int_0^T e^{-\int_0^s [d_E(x)+r]dx} ds\right] \\
&\times \left[\frac{\int_0^t e^{-\int_0^s [d_E(x)+r]dx} ds}{\int_0^T e^{-\int_0^s [d_E(x)+r]dx} ds} - \frac{\int_0^t p_C(s)^{-\zeta(1-\rho)/\rho} e^{-\frac{(\beta-r(1-\rho))}{\rho}s} ds}{\int_0^T p_C(s)^{-\zeta(1-\rho)/\rho} e^{-\frac{(\beta-r(1-\rho))}{\rho}s} ds}\right] \gtreqless 0, \\
\left.\frac{\partial q_A(0)}{\partial E_0}\right|_T &= \frac{-\int_0^T e^{-\int_0^s [d_E(x)+r]dx} ds}{\frac{\Lambda}{\rho} q_A(0)^{-\left(\frac{1+\rho}{\rho}\right)} \int_0^T p_C(s)^{-\frac{\zeta(1-\rho)}{\rho}} e^{-\frac{(\beta-r(1-\rho))}{\rho}s} ds} < 0, \\
\left.\frac{\partial X_C(t)}{\partial E_0}\right|_T &= -\zeta\Lambda \frac{q_A(0)^{-\frac{(1+\rho)}{\rho}}}{\rho} p_C(t)^{-(1-\zeta+\zeta/\rho)} e^{-\frac{(\beta-r)}{\rho}t} \left.\frac{\partial q_A(0)}{\partial E_0}\right|_T \\
&= \frac{\zeta p_C(t)^{-(1-\zeta+\zeta/\rho)} e^{-\frac{(\beta-r)}{\rho}t} \int_0^T e^{-\int_0^s [d_E(x)+r]dx} ds}{\int_0^T p_C(s)^{-\frac{\zeta(1-\rho)}{\rho}} e^{-\frac{(\beta-r(1-\rho))}{\rho}s} ds} > 0, \\
\left.\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial E_0}\right|_T &= -(1-\zeta)\Lambda \frac{q_A(0)^{-\frac{(1+\rho)}{\rho}}}{\rho} p_C(t)^{-\frac{\zeta(1-\rho)}{\rho}} e^{-\frac{(\beta-r)}{\rho}t} \left.\frac{\partial q_A(0)}{\partial E_0}\right|_T \\
&= \frac{(1-\zeta) p_C(t)^{-\frac{\zeta(1-\rho)}{\rho}} e^{-\frac{(\beta-r)}{\rho}t} \int_0^T e^{-\int_0^t [d_E(x)+r]dx} dt}{\int_0^T p_C(s)^{-\frac{\zeta(1-\rho)}{\rho}} e^{-\frac{(\beta-r(1-\rho))}{\rho}s} ds} > 0. \qquad (87)
\end{aligned}
$$

Note that the relation for the variation in wealth has the desired properties $\partial A(0)/\partial E_0|_T = 0$, and $\partial A(T)/\partial E_0|_T = 0$. Further, the wealth effect of additional skill capital δE_0 is proportional to the effect we derived earlier of an additional amount of wealth δA_0,

$$
\begin{aligned}
\left.\frac{\partial q_A(0)}{\partial E_0}\right|_T &= \left\{\int_0^T e^{-\int_0^s [d_E(x)+r]dx} ds\right\} \left.\frac{\partial q_A(0)}{\partial A_0}\right|_T, \\
\left.\frac{\partial X_C(t)}{\partial E_0}\right|_T &= \left\{\int_0^T e^{-\int_0^s [d_E(x)+r]dx} ds\right\} \left.\frac{\partial X_C(t)}{\partial A_0}\right|_T, \\
\left.\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial E_0}\right|_T &= \left\{\int_0^T e^{-\int_0^s [d_E(x)+r]dx} ds\right\} \left.\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial A_0}\right|_T. \qquad (88)
\end{aligned}
$$

Note further, that

$$
\begin{aligned}
\left.\frac{\partial f_E[.]}{\partial E_0}\right|_T &= 0, \\
\left.\frac{\partial f_H[.]}{\partial E_0}\right|_T &= 0, \\
\left.\frac{\partial Y[.]}{\partial E_0}\right|_T &= \left.\frac{\partial E(t)}{\partial E_0}\right|_T. \qquad (89)
\end{aligned}
$$

Now allow length of life T to be optimally chosen. Using (78) we have

$$
\begin{aligned}
\frac{\partial T}{\partial E_0} &= \frac{q_A(0)e^{-\int_0^T [d_E(x)+r]dx} + \left.\frac{\partial q_A(0)}{\partial E_0}\right|_T e^{-rT}\left[\left.\frac{\partial A(t)}{\partial t}\right|_{t=T} + q_{h/a}(T)\left.\frac{\partial H(t)}{\partial t}\right|_{t=T}\right]}{-\left.\frac{\partial \Im(T)}{\partial T}\right|_T} \\
&= \frac{q_A(0)e^{-\int_0^T [d_E(x)+r]dx}}{-\left.\frac{\partial \Im(T)}{\partial T}\right|_T} + \left\{\int_0^T e^{-\int_0^s [d_E(x)+r]dx} ds\right\}\frac{\partial T}{\partial A_0} > 0. \qquad (90)
\end{aligned}
$$

Using (36), we obtain the following total responses to variation in skill capital

$$
\begin{aligned}
\frac{\partial q_{e/a}(t)}{\partial E_0} &= \left.\frac{\partial q_{e/a}(t)}{\partial T}\right|_{E_0}\frac{\partial T}{\partial E_0} > 0, &\qquad \frac{\partial q_{h/a}(t)}{\partial E_0} &= \left.\frac{\partial q_{h/a}(t)}{\partial T}\right|_{E_0}\frac{\partial T}{\partial E_0} > 0, \\
\frac{\partial E(t)}{\partial E_0} &= \left.\frac{\partial E(t)}{\partial E_0}\right|_T + \left.\frac{\partial E(t)}{\partial T}\right|_{E_0}\frac{\partial T}{\partial E_0} > 0, & \frac{\partial H(t)}{\partial E_0} &= \left.\frac{\partial H(t)}{\partial T}\right|_{E_0}\frac{\partial T}{\partial E_0} > 0, \\
\frac{\partial X_E(t)}{\partial E_0} &= \left.\frac{\partial X_E(t)}{\partial T}\right|_{E_0}\frac{\partial T}{\partial E_0} > 0, & \frac{\partial X_H(t)}{\partial E_0} &= \left.\frac{\partial X_H(t)}{\partial T}\right|_{E_0}\frac{\partial T}{\partial E_0} > 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial E_0} &= \left.\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{E_0}\frac{\partial T}{\partial E_0} > 0, \\
\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial E_0} &= \left.\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{E_0}\frac{\partial T}{\partial E_0} > 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial A(t)}{\partial E_0} &= \left.\frac{\partial A(t)}{\partial E_0}\right|_T + \left.\frac{\partial A(t)}{\partial T}\right|_{E_0}\frac{\partial T}{\partial E_0} \gtreqless 0, \\
\frac{\partial q_A(0)}{\partial E_0} &= \left.\frac{\partial q_A(0)}{\partial E_0}\right|_T + \left.\frac{\partial q_A(0)}{\partial T}\right|_{E_0}\frac{\partial T}{\partial E_0} \gtreqless 0,
\end{aligned}
$$

$$
\frac{\partial X_C(t)}{\partial E_0} = -\frac{\zeta}{\rho} q_A(0)^{-\left(\frac{1+\rho}{\rho}\right)} \Lambda p_C(t)^{-(1-\zeta+\zeta/\rho)} e^{-\left(\frac{\beta-r}{\rho}\right)t}\left[\left.\frac{\partial q_A(0)}{\partial E_0}\right|_T + \left.\frac{\partial q_A(0)}{\partial T}\right|_{E_0}\frac{\partial T}{\partial E_0}\right] \gtreqless 0,
$$

$$
\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial E_0} = \left.\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial E_0}\right|_T + \left.\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{E_0}\frac{\partial T}{\partial E_0} \gtreqless 0, \qquad (91)
$$

where we have used (83).

Comparative dynamics of initial health $\partial g(t)/\partial H_0$: Again, first consider the case where T is fixed. Differentiating (74) with respect to H_0, using (66), and differentiating (75) with respect to H_0, one finds

$$\left.\frac{\partial q_{e/a}(0)}{\partial H_0}\right|_T = 0,$$
$$\left.\frac{\partial q_{h/a}(0)}{\partial H_0}\right|_T = \frac{-1}{\frac{\gamma_H}{1-\gamma_H}\int_0^T \mu_H(s) q_{h/a}(s)^{\frac{2\gamma_H-1}{1-\gamma_H}} e^{\int_0^s [2d_H(x)+r]dx} ds} < 0. \tag{92}$$

Using (54) to (74), and (92), we obtain

$$\left.\frac{\partial q_{e/a}(t)}{\partial H_0}\right|_T = 0, \ \forall t$$
$$\left.\frac{\partial q_{h/a}(t)}{\partial H_0}\right|_T = \left.\frac{\partial q_{h/a}(0)}{\partial H_0}\right|_T e^{\int_0^t [d_H(x)+r]dx} < 0,$$
$$\left.\frac{\partial E(t)}{\partial H_0}\right|_T = 0, \ \forall t$$
$$\left.\frac{\partial H(t)}{\partial H_0}\right|_T = e^{-\int_0^t d_H(x)dx}\left[1 - \frac{\int_0^t \mu_H(s) q_{h/a}(s)^{\frac{2\gamma_H-1}{1-\gamma_H}} e^{\int_0^s [2d_H(x)+r]dx} ds}{\int_0^T \mu_H(s) q_{h/a}(s)^{\frac{2\gamma_H-1}{1-\gamma_H}} e^{\int_0^s [2d_H(x)+r]dx} ds}\right] \geq 0,$$
$$\left.\frac{\partial X_E(t)}{\partial H_0}\right|_T = 0, \ \forall t$$
$$\left.\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial H_0}\right|_T = 0, \ \forall t$$
$$\left.\frac{\partial X_H(t)}{\partial H_0}\right|_T = \frac{X_H(t)}{1-\gamma_H}\frac{\left.\frac{\partial q_{h/a}(t)}{\partial H_0}\right|_T}{q_{h/a}(t)} < 0,$$

$$\left.\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial H_0}\right|_T = \frac{\tau_H(t)\left[E(t)+H(t)\right]}{1-\gamma_H}\frac{\left.\frac{\partial q_{h/a}(t)}{\partial H_0}\right|_T}{q_{h/a}(t)} < 0,$$
$$\left.\frac{\partial A(t)}{\partial H_0}\right|_T = e^{rt}\left[\int_0^T \epsilon\left[H(s), q_{h/a}(s)\right] ds\right]$$
$$\times\left[\frac{\int_0^t \epsilon\left[H(s), q_{h/a}(s)\right] ds}{\int_0^T \epsilon\left[H(s), q_{h/a}(s)\right] ds} - \frac{\int_0^t p_C(s)^{-\zeta(1-\rho)/\zeta} e^{-\frac{(\beta-r(1-\rho))}{\rho}s} ds}{\int_0^T p_C(s)^{-\zeta(1-\rho)/\zeta} e^{-\frac{(\beta-r(1-\rho))}{\rho}s} ds}\right] \gtreqless 0,$$
$$\left.\frac{\partial q_A(0)}{\partial H_0}\right|_T = \frac{-\int_0^T \epsilon\left[H(s), q_{h/a}(s)\right] ds}{q_A(0)^{\frac{-(1+\rho)}{\rho}} \frac{\Delta}{\rho}\int_0^T p_C(s)^{-\zeta(1-\rho)/\rho} e^{-\frac{(\beta-r(1-\rho))}{\rho}s} ds}$$
$$= \left\{\int_0^T \epsilon\left[H(s), q_{h/a}(s)\right] ds\right\}\left.\frac{\partial q_A(0)}{\partial A_0}\right|_T < 0, \tag{93}$$

where

$$\epsilon\left[H(s), q_{h/a}(s)\right] = \left.\frac{\partial H(s)}{\partial H_0}\right|_T - \frac{\gamma_H}{1-\gamma_H}\mu_H(s) q_{h/a}(s)^{\frac{\gamma_H}{1-\gamma_H}}\left.\frac{\partial q_{h/a}(s)}{\partial H_0}\right|_T e^{-rs} > 0, \ \forall s$$

and we have used $\partial H(s)/\partial H_0|_T > 0$ and $\partial q_{h/a}(s)/\partial H_0\big|_T < 0$ (see 93).

Further using (70) and (71) it follows that

$$\begin{aligned}
\left.\frac{\partial X_C(t)}{\partial H_0}\right|_T &= -\frac{1}{\rho}\zeta\Lambda q_A(0)^{-\frac{(1+\rho)}{\rho}} p_C(t)^{-(1-\zeta+\zeta/\rho)} e^{-\frac{\beta-r}{\rho}t} \left.\frac{\partial q_A(0)}{\partial H_0}\right|_T > 0, \\
\left.\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial H_0}\right|_T &= -\frac{1}{\rho}(1-\zeta)\Lambda q_A(0)^{-\frac{(1+\rho)}{\rho}} p_C(t)^{-\zeta(1-\rho)/\rho} e^{-\frac{\beta-r}{\rho}t} \left.\frac{\partial q_A(0)}{\partial H_0}\right|_T > 0. \qquad (94)
\end{aligned}$$

Note that the relation for the variation in the health stock has the desired properties $\partial H(0)/\partial H_0|_T = 1$, and $\partial H(T)/\partial H_0|_T = 0$, and the relation for the variation in wealth has the desired properties $\partial A(0)/\partial H_0|_T = 0$, and $\partial A(T)/\partial H_0|_T = 0$. Also note that

$$\begin{aligned}
\left.\frac{\partial f_E[.]}{\partial H_0}\right|_T &= 0, \\
\left.\frac{\partial f_H[.]}{\partial H_0}\right|_T &= \frac{\gamma_H}{1-\gamma_H}\frac{f_H[.]}{q_{h/a}(t)} \left.\frac{\partial q_{h/a}(t)}{\partial H_0}\right|_T < 0, \qquad (95)
\end{aligned}$$

so that the additional productivity $f_E[.]$ from greater health, $\partial H(t)/\partial H_0|_T > 0$, is exactly offset by the reduction in time inputs, $\partial \tau_E(t)/\partial H_0|_T < 0$, and, the additional productivity $f_H[.]$ from greater health, $\partial H(t)/\partial H_0|_T > 0$, is more than offset, $\partial f_H[.]/\partial E_0|_T < 0$, in order to ensure that length of life remains of the same duration (we assumed fixed T).

Now allow length of life T to be optimally chosen. Using (78) we have

$$\begin{aligned}
\frac{\partial T}{\partial H_0} &= \frac{\left.\frac{\partial q_A(0)}{\partial H_0}\right|_T e^{-rT} \left.\frac{\partial A(t)}{\partial t}\right|_{t=T} + \left[q_{h/a}(T)\left.\frac{\partial q_A(0)}{\partial H_0}\right|_T e^{-rT} + q_A(T)\left.\frac{\partial q_{h/a}(T)}{\partial H_0}\right|_T\right]\left.\frac{\partial H(t)}{\partial t}\right|_{t=T}}{-\left.\frac{\partial \Im(T)}{\partial T}\right|_{H_0}} \\
&= \frac{q_A(T)\left.\frac{\partial q_{h/a}(T)}{\partial H_0}\right|_T \left.\frac{\partial H(t)}{\partial t}\right|_{t=T}}{-\left.\frac{\partial \Im(T)}{\partial T}\right|_{H_0}} + \left\{\int_0^T \epsilon\left[H(s), q_{h/a}(s)\right] ds\right\}\frac{\partial T}{\partial A_0} > 0. \qquad (96)
\end{aligned}$$

Using (36), we obtain the following total responses to variation in skill capital

$$\begin{aligned}
\frac{\partial q_{e/a}(t)}{\partial H_0} &= \left.\frac{\partial q_{e/a}(t)}{\partial T}\right|_{H_0}\frac{\partial T}{\partial H_0} > 0, &\quad \frac{\partial q_{h/a}(t)}{\partial H_0} &= \left.\frac{\partial q_{h/a}(t)}{\partial H_0}\right|_T + \left.\frac{\partial q_{h/a}(t)}{\partial T}\right|_{H_0}\frac{\partial T}{\partial H_0} \gtreqless 0, \\
\frac{\partial E(t)}{\partial H_0} &= \left.\frac{\partial E(t)}{\partial T}\right|_{H_0}\frac{\partial T}{\partial H_0} > 0, &\quad \frac{\partial H(t)}{\partial H_0} &= \left.\frac{\partial H(t)}{\partial H_0}\right|_T + \left.\frac{\partial H(t)}{\partial T}\right|_{H_0}\frac{\partial T}{\partial H_0} > 0, \\
\frac{\partial X_E(t)}{\partial H_0} &= \left.\frac{\partial X_E(t)}{\partial T}\right|_{H_0}\frac{\partial T}{\partial H_0} > 0, &\quad \frac{\partial X_H(t)}{\partial H_0} &= \left.\frac{\partial X_H(t)}{\partial H_0}\right|_T + \left.\frac{\partial X_H(t)}{\partial T}\right|_{H_0}\frac{\partial T}{\partial H_0} \gtreqless 0,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial H_0} &= \left.\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{H_0}\frac{\partial T}{\partial H_0} > 0, \\
\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial H_0} &= \left.\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial H_0}\right|_T + \left.\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{H_0}\frac{\partial T}{\partial H_0} \gtreqless 0,
\end{aligned}$$

$$\frac{\partial A(t)}{\partial H_0} = \left.\frac{\partial A(t)}{\partial H_0}\right|_T + \left.\frac{\partial A(t)}{\partial T}\right|_{H_0} \frac{\partial T}{\partial H_0} \gtreqless 0,$$

$$\frac{\partial q_A(0)}{\partial H_0} = \left.\frac{\partial q_A(0)}{\partial H_0}\right|_T + \left.\frac{\partial q_A(0)}{\partial T}\right|_{H_0} \frac{\partial T}{\partial H_0} \gtreqless 0,$$

$$\frac{\partial X_C(t)}{\partial H_0} = -\frac{\zeta}{\rho} q_A(0)^{-\left(\frac{1+\rho}{\rho}\right)} \Lambda p_C(t)^{-(1-\zeta+\zeta/\rho)} e^{-\left(\frac{\beta-r}{\rho}\right)t} \left[\left.\frac{\partial q_A(0)}{\partial H_0}\right|_T + \left.\frac{\partial q_A(0)}{\partial T}\right|_{H_0} \frac{\partial T}{\partial H_0} \right] \gtreqless 0,$$

$$\frac{\partial L(t)\,[E(t)+H(t)]}{\partial H_0} = \left.\frac{\partial L(t)\,[E(t)+H(t)]}{\partial H_0}\right|_T + \left.\frac{\partial L(t)\,[E(t)+H(t)]}{\partial T}\right|_{H_0} \frac{\partial T}{\partial H_0} \gtreqless 0, \quad (97)$$

where we have used (83).

Skill and health productivity: The comparative dynamics for the skill productivity factor $\mu_E(x)$ and the health productivity factor $\mu_H(x)$ are summarized in Table 4.

Table 4: Comparative dynamic effects of the generalized skill productivity factor $\mu_E(t)$ and the generalized health productivity factor $\mu_H(t)$ on the state and co-state functions, control functions and the parameter T.

	$\mu_E(t)$		$\mu_H(t)$	
Function	T fixed	T free	T fixed	T free
$E(t)$	> 0	> 0	0	> 0
$q_{e/a}(t)$	0	> 0	0	> 0
$X_E(t)$	> 0	> 0	0	> 0
$\tau_E(t)\,[E(t)+H(t)]$	> 0	> 0	0	> 0
$H(t)$	0	> 0	≥ 0	> 0
$q_{h/a}(t)$	0	> 0	< 0	+/-
$X_H(t)$	0	> 0	+/-	+/-
$\tau_H(t)\,[E(t)+H(t)]$	0	> 0	+/-	+/-
$A(t)$	+/-	+/-	+/-	+/-
$q_A(0)$	< 0	$< 0^\dagger$	< 0	$< 0^\dagger$
$X_C(t)$	> 0	$> 0^\dagger$	> 0	$> 0^\dagger$
$L(t)\,[E(t)+H(t)]$	> 0	$> 0^\dagger$	> 0	$> 0^\dagger$
T	n/a	> 0	n/a	> 0

Notes: 0 is used to denote 'not affected', $+/-$ is used to denote that the sign is 'undetermined', n/a stands for 'not applicable', and † is used to denote that the 'sign holds under the plausible assumption that the wealth effect dominates the effect of life extension'. This is consistent with the empirical finding (Imbens, Rubin and Sacerdote 2001; Juster et al. 2006; Brown, Coile, and Weisbenner, 2010) that additional wealth leads to higher consumption, even though the horizon over which consumption takes place is extended.

Comparative dynamics of skill productivity $\partial g(t)/\partial \mu_E(x)$: Consider the case where T is fixed. Differentiating (74) with respect to $\mu_E(x)$, using (66), and differentiating (75) with respect to $\mu_E(x)$, one finds

$$\begin{aligned}\left.\frac{\partial q_{e/a}(0)}{\partial \mu_E(x)}\right|_T &= 0,\\ \left.\frac{\partial q_{h/a}(0)}{\partial \mu_E(x)}\right|_T &= 0.\end{aligned} \tag{98}$$

Using (54) to (67) we obtain

$$\begin{aligned}\left.\frac{\partial q_{e/a}(t)}{\partial \mu_E(x)}\right|_T &= 0,\ \forall t\\ \left.\frac{\partial q_{h/a}(t)}{\partial \mu_E(x)}\right|_T &= 0,\ \forall t\\ \left.\frac{\partial E(t)}{\partial \mu_E(x)}\right|_T &= q_{e/a}(x)^{\frac{\gamma_E}{1-\gamma_E}} e^{-\int_x^t d_E(u)du} > 0,\ \text{for } t \geq x,\\ \left.\frac{\partial H(t)}{\partial \mu_E(x)}\right|_T &= 0,\ \forall t\\ \left.\frac{\partial X_E(t)}{\partial \mu_E(x)}\right|_T &= \frac{\beta_E}{p_E(t)} q_{e/a}(t)^{\frac{1}{1-\gamma_E}} \delta(t-x) > 0,\ \forall t\\ \left.\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial \mu_E(x)}\right|_T &= \alpha_E q_{e/a}(t)^{\frac{1}{1-\gamma_E}} \delta(t-x) > 0,\ \forall t\\ \left.\frac{\partial X_H(t)}{\partial \mu_E(x)}\right|_T &= 0,\ \forall t\\ \left.\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial \mu_E(x)}\right|_T &= 0,\ \forall t\end{aligned} \tag{99}$$

where $\delta(t-x)$ is the Dirac Delta function, which is zero everywhere except for $t = x$ and has a total area of 1 (it is the continuous-time equivalent of the discrete Kronecker delta function).

Differentiating (73) with respect to $\mu_E(x)$, we have (for $t \geq x$)

$$\begin{aligned}\left.\frac{\partial A(t)}{\partial \mu_E(x)}\right|_T &= e^{rt} q_{e/a}(x)^{\frac{\gamma_E}{1-\gamma_E}} \left\{ \left[\int_0^t e^{-rs} e^{-\int_x^s d_E(u)du} ds - \gamma_E q_{e/a}(x) e^{-rx} \right] - \right.\\ &\left. \left[\int_0^T e^{-rs} e^{-\int_x^s d_E(u)du} ds - \gamma_E q_{e/a}(x) e^{-rx} \right] \frac{\int_0^t p_C(s)^{-\zeta(1-\rho)/\rho} e^{-\frac{(\beta - r(1-\rho))}{\rho} s} ds}{\int_0^T p_C(s)^{-\zeta(1-\rho)/\rho} e^{-\frac{(\beta - r(1-\rho))}{\rho} s} ds} \right\} \gtreqless 0,\end{aligned} \tag{100}$$

$$\left.\frac{\partial q_A(0)}{\partial \mu_E(x)}\right|_T = \frac{\gamma_E q_{e/a}(x)^{\frac{1}{1-\gamma_E}} e^{-rx} - q_{e/a}(x)^{\frac{\gamma_E}{1-\gamma_E}} \int_0^T e^{-rs} e^{\int_x^s d_E(u)du} ds}{q_A(0)^{\frac{-(1+\rho)}{\rho}} \frac{\Lambda}{\rho} \int_0^T p_C(s)^{-\zeta(1-\rho)/\rho} e^{-\frac{(\beta - r(1-\rho))}{\rho} s} ds} < 0, \tag{101}$$

where, in signing the term, we have assumed that the transient effect (first term in the numerator) is dominated by the permanent effect (second term in the numerator).

Further using (70) and (71) it follows that

$$\left.\frac{\partial X_C(t)}{\partial \mu_E(x)}\right|_T = -\frac{1}{\rho}\zeta\Lambda q_A(0)^{-\frac{(1+\rho)}{\rho}} p_C(t)^{-(1-\zeta+\zeta/\rho)} e^{-\frac{\beta-r}{\rho}t} \left.\frac{\partial q_A(0)}{\partial \mu_E(x)}\right|_T > 0, \tag{102}$$

$$\left.\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial \mu_E(x)}\right|_T = -\frac{1}{\rho}(1-\zeta)\Lambda q_A(0)^{-\frac{(1+\rho)}{\rho}} p_C(t)^{-\zeta(1-\rho)/\rho} e^{-\frac{\beta-r}{\rho}t} \left.\frac{\partial q_A(0)}{\partial \mu_E(x)}\right|_T > 0. \tag{103}$$

Now allow length of life T to be optimally chosen. Using (78) we have

$$\frac{\partial T}{\partial \mu_E(x)} = \frac{q_A(T) \left.\frac{\partial E(T)}{\partial \mu_E(x)}\right|_T + \left.\frac{\partial q_A(0)}{\partial \mu_E(x)}\right|_T e^{-rT}\left[\left.\frac{\partial A(t)}{\partial t}\right|_{t=T} + q_{h/a}(T)\left.\frac{\partial H(t)}{\partial t}\right|_{t=T}\right]}{-\left.\frac{\partial \Im(T)}{\partial T}\right|_{\mu_E(x)}} > 0. \tag{104}$$

Using (36), we obtain the following total responses to variation in the generalized productivity of skill investment, $\mu_E(x)$:

$$\frac{\partial q_{e/a}(t)}{\partial \mu_E(x)} = \left.\frac{\partial q_{e/a}(t)}{\partial T}\right|_{\mu_E(x)} \frac{\partial T}{\partial \mu_E(x)} > 0, \quad \frac{\partial q_{h/a}(t)}{\partial \mu_E(x)} = \left.\frac{\partial q_{h/a}(t)}{\partial T}\right|_{\mu_E(x)} \frac{\partial T}{\partial \mu_E(x)} > 0,$$

$$\frac{\partial E(t)}{\partial \mu_E(x)} = \left.\frac{\partial E(t)}{\partial \mu_E(x)}\right|_T + \left.\frac{\partial E(t)}{\partial T}\right|_{\mu_E(x)} \frac{\partial T}{\partial \mu_E(x)} > 0, \quad \frac{\partial H(t)}{\partial \mu_E(x)} = \left.\frac{\partial H(t)}{\partial T}\right|_{\mu_E(x)} \frac{\partial T}{\partial \mu_E(x)} > 0,$$

$$\frac{\partial X_E(t)}{\partial \mu_E(x)} = \left.\frac{\partial X_E(t)}{\partial \mu_E(x)}\right|_T + \left.\frac{\partial X_E(t)}{\partial T}\right|_{\mu_E(x)} \frac{\partial T}{\partial \mu_E(x)} > 0, \quad \frac{\partial X_H(t)}{\partial \mu_E(x)} = \left.\frac{\partial X_H(t)}{\partial T}\right|_{\mu_E(x)} \frac{\partial T}{\partial \mu_E(x)} > 0,$$

$$\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial \mu_E(x)} = \left.\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial \mu_E(x)}\right|_T + \left.\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{\mu_E(x)} \frac{\partial T}{\partial \mu_E(x)} > 0,$$

$$\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial \mu_E(x)} = \left.\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{\mu_E(x)} \frac{\partial T}{\partial \mu_E(x)} > 0,$$

$$\frac{\partial A(t)}{\partial \mu_E(x)} = \left.\frac{\partial A(t)}{\partial \mu_E(x)}\right|_T + \left.\frac{\partial A(t)}{\partial T}\right|_{\mu_E(x)} \frac{\partial T}{\partial \mu_E(x)} \gtreqqless 0,$$

$$\frac{\partial q_A(0)}{\partial \mu_E(x)} = \left.\frac{\partial q_A(0)}{\partial \mu_E(x)}\right|_T + \left.\frac{\partial q_A(0)}{\partial T}\right|_{\mu_E(x)} \frac{\partial T}{\partial \mu_E(x)} \gtreqqless 0,$$

$$\frac{\partial X_C(t)}{\partial \mu_E(x)} = \left.\frac{\partial X_C(t)}{\partial \mu_E(x)}\right|_T + \left.\frac{\partial X_C(t)}{\partial T}\right|_{\mu_E(x)} \frac{\partial T}{\partial \mu_E(x)} \gtreqqless 0,$$

$$\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial \mu_E(x)} = \left.\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial \mu_E(x)}\right|_T + \left.\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{\mu_E(x)} \frac{\partial T}{\partial \mu_E(x)} \gtreqqless 0, \tag{105}$$

where we have used (83).

Comparative dynamics of health productivity $\partial g(t)/\partial \mu_H(x)$: Consider the case where T is fixed. Differentiating (74) with respect to $\mu_H(x)$, using (66), and differentiating (75) with respect to $\mu_H(x)$, one finds

$$\begin{aligned}
\left.\frac{\partial q_{e/a}(0)}{\partial \mu_H(x)}\right|_T &= 0,\\
\left.\frac{\partial q_{h/a}(0)}{\partial \mu_H(x)}\right|_T &= \frac{-q_{h/a}(x)^{\frac{\gamma_H}{1-\gamma_H}} e^{\int_0^x d_H(u)du}}{\frac{\gamma_H}{1-\gamma_H}\int_0^T \mu_H(s) q_{h/a}(s)^{\frac{2\gamma_H-1}{1-\gamma_H}} e^{\int_0^s [2d_H(u)+r]du} ds} < 0. \qquad (106)
\end{aligned}$$

Using (54) to (67), and (106), we obtain

$$\begin{aligned}
\left.\frac{\partial q_{e/a}(t)}{\partial \mu_H(x)}\right|_T &= 0, \ \forall t\\
\left.\frac{\partial q_{h/a}(t)}{\partial \mu_H(x)}\right|_T &= \left.\frac{\partial q_{h/a}(0)}{\partial \mu_H(x)}\right|_T e^{\int_0^t [d_H(u)+r]du} < 0, \ \forall t\\
\left.\frac{\partial E(t)}{\partial \mu_H(x)}\right|_T &= 0, \ \forall t\\
\left.\frac{\partial H(t)}{\partial \mu_H(x)}\right|_T &= q_{h/a}(x)^{\frac{\gamma_H}{1-\gamma_H}} e^{-\int_x^t d_H(u)du} \times \\
&\quad \left[1 - \frac{\int_0^t \mu_H(s) q_{h/a}(s)^{\frac{2\gamma_H-1}{1-\gamma_H}} e^{\int_0^s [2d_H(u)+r]du} ds}{\int_0^T \mu_H(s) q_{h/a}(s)^{\frac{2\gamma_H-1}{1-\gamma_H}} e^{\int_0^s [2d_H(u)+r]du} ds}\right] \geq 0, \ \text{for } t \geq x,\\
\left.\frac{\partial X_E(t)}{\partial \mu_H(x)}\right|_T &= 0, \ \forall t\\
\left.\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial \mu_H(x)}\right|_T &= 0, \ \forall t
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial X_H(t)}{\partial \mu_H(x)}\right|_T &= \frac{\beta_H}{p_H(t)} q_{h/a}(t)^{\frac{1}{1-\gamma_H}} \delta(x-t) + \\
&\quad \frac{\beta_H \mu_H(t)}{p_H(t)} \frac{1}{1-\gamma_H} q_{h/a}(t)^{\frac{\gamma_H}{1-\gamma_H}} \left.\frac{\partial q_{h/a}(t)}{\partial \mu_H(x)}\right|_T < 0 \text{ for } t \neq x,\\
\left.\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial \mu_H(x)}\right|_T &= \alpha_H q_{h/a}(t)^{\frac{1}{1-\gamma_H}} \delta(x-t) + \\
&\quad \alpha_H \mu_H(t) \frac{1}{1-\gamma_H} q_{h/a}(t)^{\frac{\gamma_H}{1-\gamma_H}} \left.\frac{\partial q_{h/a}(t)}{\partial \mu_H(x)}\right|_T < 0 \text{ for } t \neq x. \qquad (107)
\end{aligned}$$

Differentiating (73) with respect to $\mu_H(x)$, we have

$$\left.\frac{\partial A(t)}{\partial \mu_H(x)}\right|_T = e^{rt}\left\{\chi(t,x) - \chi(T,x)\frac{\int_0^t p_C(s)^{-\zeta(1-\rho)/\rho} e^{-\frac{(\beta-r(1-\rho))}{\rho}s}ds}{\int_0^T p_C(s)^{-\zeta(1-\rho)/\rho} e^{-\frac{(\beta-r(1-\rho))}{\rho}s}ds}\right\} \gtreqless 0, \tag{108}$$

$$\left.\frac{\partial q_A(0)}{\partial \mu_H(x)}\right|_T = \frac{-\chi(T,x)}{q_A(0)^{\frac{-(1+\rho)}{\rho}}\frac{\Lambda}{\rho}\int_0^T p_C(s)^{-\zeta(1-\rho)/\rho} e^{-\frac{(\beta-r(1-\rho))}{\rho}s}ds} < 0, \tag{109}$$

where

$$\chi(t,x) = -e^{-rx}\gamma_H q_{h/a}(x)^{\frac{1}{1-\gamma_H}} + \int_0^t \left[\left.\frac{\partial H(s)}{\partial \mu_H(x)}\right|_T - \frac{\gamma_H}{1-\gamma_H}\mu_H(s) q_{h/a}(s)^{\frac{\gamma_H}{1-\gamma_H}}\left.\frac{\partial q_{h/a}(s)}{\partial \mu_H(x)}\right|_T\right] e^{-rs}ds > 0, \tag{110}$$

and, in signing the terms, we have assumed once more that permanent effects dominate transient effects.

Further using (70) and (71) it follows that

$$\left.\frac{\partial X_C(t)}{\partial \mu_H(x)}\right|_T = -\frac{1}{\rho}\zeta\Lambda q_A(0)^{-\frac{(1+\rho)}{\rho}} p_C(t)^{-(1-\zeta+\zeta/\rho)} e^{-\frac{\beta-r}{\rho}t}\left.\frac{\partial q_A(0)}{\partial \mu_H(x)}\right|_T > 0, \tag{111}$$

$$\left.\frac{\partial L(t)\left[E(t)+H(t)\right]}{\partial \mu_H(x)}\right|_T = -\frac{1}{\rho}(1-\zeta)\Lambda q_A(0)^{-\frac{(1+\rho)}{\rho}} p_C(t)^{-\zeta(1-\rho)/\rho} e^{-\frac{\beta-r}{\rho}t}\left.\frac{\partial q_A(0)}{\partial \mu_H(x)}\right|_T > 0. \tag{112}$$

Now allow length of life T to be optimally chosen. Using (78) we have

$$\frac{\partial T}{\partial \mu_H(x)} = \frac{\left.\frac{\partial q_A(T)}{\partial \mu_H(x)}\right|_T \left.\frac{\partial A(t)}{\partial t}\right|_{t=T} + \left[q_{h/a}(T)\left.\frac{\partial q_A(T)}{\partial \mu_H(x)}\right|_T + q_A(T)\left.\frac{\partial q_{h/a}(T)}{\partial \mu_H(x)}\right|_T\right]\left.\frac{\partial H(t)}{\partial t}\right|_{t=T}}{-\left.\frac{\partial \Im(T)}{\partial T}\right|_{\mu_H(x)}} > 0. \tag{113}$$

Using (36), we obtain the following total responses to variation in the generalized productivity of health investment, $\mu_H(x)$:

$$\frac{\partial q_{e/a}(t)}{\partial \mu_H(x)} = \left.\frac{\partial q_{e/a}(t)}{\partial T}\right|_{\mu_H(x)}\frac{\partial T}{\partial \mu_H(x)} > 0, \quad \frac{\partial q_{h/a}(t)}{\partial \mu_H(x)} = \left.\frac{\partial q_{h/a}(t)}{\partial \mu_H(x)}\right|_T + \left.\frac{\partial q_{h/a}(t)}{\partial T}\right|_{\mu_H(x)}\frac{\partial T}{\partial \mu_H(x)} \gtreqless 0,$$

$$\frac{\partial E(t)}{\partial \mu_H(x)} = \left.\frac{\partial E(t)}{\partial T}\right|_{\mu_H(x)}\frac{\partial T}{\partial \mu_H(x)} > 0, \quad \frac{\partial H(t)}{\partial \mu_H(x)} = \left.\frac{\partial H(t)}{\partial \mu_H(x)}\right|_T + \left.\frac{\partial H(t)}{\partial T}\right|_{\mu_H(x)}\frac{\partial T}{\partial \mu_H(x)} > 0,$$

$$\frac{\partial X_E(t)}{\partial \mu_H(x)} = \left.\frac{\partial X_E(t)}{\partial T}\right|_{\mu_H(x)}\frac{\partial T}{\partial \mu_H(x)} > 0, \quad \frac{\partial X_H(t)}{\partial \mu_H(x)} = \left.\frac{\partial X_H(t)}{\partial \mu_H(x)}\right|_T + \left.\frac{\partial X_H(t)}{\partial T}\right|_{\mu_H(x)}\frac{\partial T}{\partial \mu_H(x)} > 0,$$

$$\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial \mu_H(x)} = \left.\frac{\partial \tau_E(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{\mu_H(x)}\frac{\partial T}{\partial \mu_H(x)} > 0,$$

$$\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial \mu_H(x)} = \left.\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial \mu_H(x)}\right|_T + \left.\frac{\partial \tau_H(t)\left[E(t)+H(t)\right]}{\partial T}\right|_{\mu_H(x)}\frac{\partial T}{\partial \mu_H(x)} > 0,$$

$$\frac{\partial A(t)}{\partial \mu_H(x)} = \left.\frac{\partial A(t)}{\partial \mu_H(x)}\right|_T + \left.\frac{\partial A(t)}{\partial T}\right|_{\mu_H(x)} \frac{\partial T}{\partial \mu_H(x)} \gtreqless 0,$$

$$\frac{\partial q_A(0)}{\partial \mu_H(x)} = \left.\frac{\partial q_A(0)}{\partial \mu_H(x)}\right|_T + \left.\frac{\partial q_A(0)}{\partial T}\right|_{\mu_H(x)} \frac{\partial T}{\partial \mu_H(x)} \gtreqless 0,$$

$$\frac{\partial X_C(t)}{\partial \mu_H(x)} = \left.\frac{\partial X_C(t)}{\partial \mu_H(x)}\right|_T + \left.\frac{\partial X_C(t)}{\partial T}\right|_{\mu_H(x)} \frac{\partial T}{\partial \mu_H(x)} \gtreqless 0,$$

$$\frac{\partial L(t)\,[E(t)+H(t)]}{\partial \mu_H(x)} = \left.\frac{\partial L(t)\,[E(t)+H(t)]}{\partial \mu_H(x)}\right|_T + \left.\frac{\partial L(t)\,[E(t)+H(t)]}{\partial T}\right|_{\mu_H(x)} \frac{\partial T}{\partial \mu_H(x)} \gtreqless 0, \quad (114)$$

where we have used (83).

A.6 Generalized Heckman result: Absent ability to extend life T, associations between wealth, skill and health are absent or small.

This prediction is true also for the general model. In both the general and the simpler model, the end condition applies that end of life occurs at $t = T$ at the minimum health level $H(T) = H_{\min}$. Hence, even though in the general model additional resources in the form of wealth, skill, or health, may lead to an initial increase in the relative marginal value of health $q_{h/a}(t)$ (see 25) and therefore greater health investment and greater health, for fixed length of life T this needs to be compensated by eventual lower health investment in order for health to reach $H_{\min}$ at $t = T$. The response to additional resources of health investment and thereby health is therefore muted.

While skill may be more responsive to additional resources, as its terminal level $E(T)$ is allowed to be free, the response to wealth of skill investment and skill is also muted due to strong complementarity between skill and health: the initial benefits derived from higher levels of health (earnings, self-productivity, and dynamic complementarity) are offset by subsequent lower benefits from reduced health. Moreover, one of the key drivers of skill-capital investment is the horizon (longevity). This important pathway is shut down when forcing length of life T to be fixed. As a result, there are no strong associations between wealth, skill, and health for fixed T, and $\partial g(t)/\partial Z_0|_T$, the first term on the RHS of (36), is generally small for variation δZ_0 in any model parameter of interest.

A.7 Generalized Ehrlich and Chuma result: Wealthy, skilled, and healthy individuals live longer.

Individuals optimally choose longevity T such that the marginal value of life extension is zero at this age, $\Im(T) = 0$ (see 14),

$$\Im(T) = U(T)e^{-\beta T} + q_H(T) \left.\frac{\partial H}{\partial t}\right|_{t=T} + q_A(T) \left.\frac{\partial A}{\partial t}\right|_{t=T} = 0, \quad (115)$$

where we have used the transversality condition $q_E(T) = 0$. As the expression shows, the marginal benefit of extending life consists of the additional utility from consumption and effective leisure, and the marginal costs consist of the increasingly binding wealth and health constraints, due to declining wealth and declining health near the end of life.[37] In particular, health is increasingly constraining relative to wealth as the marginal value of wealth $q_A(t) = q_A(0)e^{-rt}$ declines with age while the relative marginal value of health increases with age $q_{h/a}(t) = q_H(t)/q_A(t)$ (i.e. even if $q_H(t)$ declines [but more likely, it increases] it does so less rapidly than does $q_A(t)$). In addition, declining health reduces utility $U(t)$ and thereby the marginal benefit of life extension.

The conditions (81), (82) and (115) for optimal length of life do not depend on the characteristics of the simpler model. They also apply to the general model. The lifecycle trajectories of $A(t)$ and $H(t)$ are similar in the general and simpler model – in particular in both models health and assets decline towards the end of life. Thus, in order to establish proof, using (82), we need only establish that $\partial E(T)/\partial Z_0|_T > 0$, $\partial q_A(T)/\partial Z_0|_T < 0$, and $\partial q_H(T)/\partial Z_0|_T < 0$, for $Z_0 = \{A_0, E_0, H_0\}$. From Appendix A.8 follows $\partial E(T)/\partial Z_0|_T > 0$, for $Z_0 = \{A_0, E_0, H_0\}$. By the assumption of diminishing returns to wealth, we have $\partial q_A(T)/\partial A(0)|_T = e^{-rT}\, \partial q_A(0)/\partial A(0)|_T < 0$ (wealth increases life time utility but at a diminishing rate).

For the remainder of the proof we follow the reasoning of the generalized Heckman result (section A.6). For the simple model we have $\partial q_A(T)/\partial E(0)|_T < 0$ and $\partial q_A(T)/\partial H(0)|_T < 0$, which for the general model plausibly holds as well, as follows. For fixed T any additional investment in health early in life, as a result of the additional resources δE_0 or δH_0, needs to be compensated by reduced investment later in life for health to reach $H_{\min}$ at $t = T$. Hence, we expect $\partial q_{h/a}(t)/\partial Z_0\big|_T$ to be positive up to some $t = t^{\dagger}$, and negative afterwards (see also Galama and Van Kippersluis, 2015). In particular, $\partial q_{h/a}(T)/\partial Z_0\big|_T < 0$. This also affects decisions regarding investment in skill and the response in terms of skill and health investment is muted for fixed T (see generalized Heckman result, section A.6). Since in aggregate not much additional investment is made (positive and negative variations in investment balance out), the additional resources can only be spend on consumption. This would be associated (see 15) with a reduced marginal value of wealth at any age. Therefore, $\partial q_A(T)/\partial E(0)|_T < 0$ and $\partial q_A(T)/\partial H(0)|_T < 0$. Last, we need to establish that $\partial q_H(T)/\partial Z_0|_T < 0$, for $Z_0 = \{A_0, E_0, H_0\}$. Now, $\partial q_H(T)/\partial Z_0|_T = q_A(T)\, \partial q_{h/a}(T)/\partial Z_0\big|_T + q_A(T)^{-1}\, \partial q_A(T)/\partial Z_0|_T$. Both terms on the RHS are negative as discussed above. Q.E.D.

Thus we have established that the prediction that wealthy, skilled, and healthy individuals live longer also plausibly holds in the general model.

[37] Both $\partial H(t)/\partial t|_{t=T}$ and $\partial A(t)/\partial t|_{t=T}$ are negative since health declines near the end of life as it approaches $H_{\min}$ from above, and assets decline near the end of life in absence of a very strong bequest motive.

A.8 Miscellaneous proofs

Wealthy and healthy individuals value skill more, invest more in skill, and are more skilled at every age. Individuals with more endowed skill are more skilled at every age, but potentially value skill less.

Wealthy and skilled individuals value health more, invest more in health, and are healthier at every age. Individuals with more endowed health are healthier at every age, but potentially value health less.

The first four rows of Table 1 show that positive variations in the form of endowed wealth, skill, or health, lead to a higher marginal value of skill $q_{e/a}(t)$, higher levels of investment inputs $X_E(t)$, $\tau_E(t)[E(t)+H(t)]$, and greater skill $E(t)$ (T free). A longer horizon increases the return to investment in skill, such that wealthy individuals value skill more. As a result they invest more in skill and are more skilled at every age $\partial E(t)/\partial A_0 > 0 \quad \forall t$. Endowments in skill and health are also forms of wealth, so that similar reasoning can be applied here.

In the general model wealthy and healthy individuals also value skill more. Investment in skill is one margin of adjustment individuals can choose with several benefits: skill capital increases earnings $\partial Y/\partial E > 0$, the efficiency of skill production $\partial f_E/\partial E > 0$, and the efficiency of health production $\partial f_H/\partial E > 0$, and skill capital extends the horizon (generalized Ehrlich and Chuma result, section A.7), thereby increasing the return on skill-capital investment (see 19). Wealth provides additional resources that can be devoted to skill investment and so does health, but health also raises the various benefits of skill capital as $\partial^2 Y/\partial E\partial H > 0$ (skill raises wages and health increases time devoted to work), $\partial^2 f_E/\partial E\partial H > 0$ (both skill and health raise the productivity of skill formation), and plausibly $\partial^2 f_H/\partial E\partial H > 0$. Thus, both wealth and health increase skill investment and skill. In the general model the effect of health on skill is plausibly even larger than in the simpler model, given the strong complementarities of health and skill in earnings, and in the production of skill and health.

Whether additional skill increases skill investment is less clear. Since in the general model $\partial^2 Y/\partial E^2 < 0$, $\partial^2 f_E/\partial E^2 < 0$, and $\partial^2 f_H/\partial E^2 < 0$, the various benefits of skill capital are decreasing in endowed skill, providing incentives to reduce skill-capital investment. Nonetheless, starting out with higher skill, under standard economic assumptions regarding the functional forms of the utility and production functions and our assumed complementarity between skill and health (Y, f_E and f_H), skill investment will not be reduced to such an extent that skill is eventually lower for individuals who started out with a greater endowment of skill.

In sum, the only difference with the simpler model is that skilled individuals could potentially value skill less $\partial q_{e/a}(t)/\partial E_0 < 0$, and therefore invest less, but still have greater skill at every age.

The fifth to eight rows of Table 1 show that positive variation in the form of greater endowed wealth and skill, lead to a higher marginal value of health $q_{h/a}(t)$, higher levels of investment inputs $X_H(t)$, $\tau_H(t)[E(t) + H(t)]$, and greater health $H(t)$ (T free). Yet, while endowed health does lead to greater health at every age $\partial H(t)/\partial H_0 > 0 \quad \forall t$, it does not unambiguously lead to a higher marginal value of health $q_{h/a}(t)$, and thereby higher levels of investment $X_H(t)$, $\tau_H(t)[E(t)+H(t)]$. The easiest way to understand this is for fixed length of life. In contrast to skill capital, where the terminal value $E(T)$ is unconstrained, the terminal value of health is $H_{\min}$. If the horizon T is fixed, additional health $H_0 + \delta H_0$ needs to be offset by lower health investment throughout life in order to reach $H_{\min}$ at $t = T$. For free length of life T, if the increase in length of life $\partial T/\partial H_0$ is sufficiently large, health investment is higher throughout life. If it is small, health investment is lower throughout life.

In the general model, this prediction plausibly applies too. Wealthy and skilled individuals value health more for its many benefits: health provides utility $\partial U/\partial H > 0$, increases earnings $\partial Y/\partial H > 0$, the efficiency of skill production $\partial f_E/\partial H > 0$, and potentially the efficiency of health production $\partial f_H/\partial H > 0$, and health capital extends the horizon (generalized Ehrlich and Chuma result, section A.7), thereby increasing the return on health-capital investment (see 24). Wealth provides additional resources that can be devoted to health investment and so does skill, but skill also raises the various benefits of health capital as $\partial^2 Y/\partial E\partial H > 0$ (skill raises wages and health increases time devoted to work), $\partial^2 f_E/\partial E\partial H > 0$ (both skill and health raise the productivity of skill formation), and plausibly $\partial^2 f_H/\partial E\partial H > 0$. These effects are plausibly larger in the general model than in the simpler model due to strong complementarity between skill and health in earnings and in the production of skill and health. Thus, both wealth and skill increase health investment and health.[38]

Similar to the discussion for skill, for greater endowed health, the demand for health investment is reduced since in the general model the various benefits of health capital are decreasing in endowed health. Nonetheless, starting out with higher health, and for the same reasons mentioned for skill, health investment will not be reduced to such an extent that health is eventually lower when starting out with a greater endowment. Thus endowed health also leads to greater health at every age. Further, health extends the horizon, thereby increasing the return on health-capital investment, so that health investment may be higher at every age, in particular if endowed health enables substantial life extension.

In sum, analogous to the case for skill, healthy individuals could potentially value health less $\partial q_{h/a}(t)/\partial H_0 < 0$ in both the simpler and the general model. This scenario seems plausible for health since health is more constrained than skill (the terminal value of health is fixed at $H_{\min}$ and, unlike skill, health potentially lowers the efficiency of the

[38] If these additional resources can only moderately extend life then any initial higher levels of health investment may have to be somewhat offset by subsequent lower investment for health to reach $H_{\min}$ (as this case more closely resembles that of fixed T). Thus, for the wealthy and skilled, health is higher and in aggregate health investment is higher, but later in the lifecycle investment may be reduced.

health-production process $\partial f_H/\partial H < 0$).

A.9 Prediction 2: Longevity gains generate health and economic inequality.

The discussion for prediction 2 in section 3.2.2 relied on the simpler model only in establishing that $\partial E(t)/\partial T|_{Z_0} > 0$ and $\partial H(t)/\partial T|_{Z_0} > 0$, for $Z_0 = \{A_0, E_0, H_0, \mu_E(t), \mu_H(t)\}$. If we can show these also hold for the general model, the proof is completed. While we cannot provide a formal proof, we can invoke a simple heuristic argument based on (36). From the generalized Heckman result (see section A.6) we know that $\partial E(t)/\partial Z_0|_T$ and $\partial H(t)/\partial Z_0|_T$ are small. So that $\partial E(t)/\partial Z_0 \approx \partial E(t)/\partial T|_{Z_0} (\partial T/\partial Z_0)$ and $\partial H(t)/\partial Z_0 \approx \partial H(t)/\partial T|_{Z_0} (\partial T/\partial Z_0)$. From the generalized Ehrlich and Chuma result (see section A.7) we have $\partial T/\partial Z_0 > 0$, for $Z_0 = \{A_0, E_0, H_0, \mu_E(t), \mu_H(t)\}$ and from section A.8 we have $\partial E(t)/\partial Z_0 > 0$ and $\partial H(t)/\partial Z_0 > 0$, for $Z_0 = \{A_0, E_0, H_0, \mu_E(t), \mu_H(t)\}$. Thus, if these predictions hold, we find $\partial E(t)/\partial T|_{Z_0} > 0$ and $\partial H(t)/\partial T|_{Z_0} > 0$, for $Z_0 = \{A_0, E_0, H_0, \mu_E(t), \mu_H(t)\}$, and $\forall t$. Q.E.D.

A.10 Prediction 3: Complementarity effects reinforce associations between wealth, skill, health, and technology

In many cases, variations in two (or more) parameters that affect longevity, reinforce each other. To see this, differentiate (36) with respect to an additional generic variation δW_0 in an initial condition or model parameter, to obtain:

$$\frac{\partial^2 g(t)}{\partial Z_0 \partial W_0} = \left.\frac{\partial^2 g(t)}{\partial Z_0 \partial W_0}\right|_T + \left.\frac{\partial^2 g(t)}{\partial T \partial W_0}\right|_{Z_0} \frac{\partial T}{\partial Z_0} + \left.\frac{\partial g(t)}{\partial T}\right|_{Z_0} \frac{\partial^2 T}{\partial Z_0 \partial W_0}. \tag{116}$$

The first term is small (for fixed T; see generalized Heckman result, section A.6). The second term increases with the extent of life extension $\partial T/\partial Z_0$. If $\partial^2 g(t)/\partial T \partial W_0\big|_{Z_0} > 0$, then there is complementarity, and variation in Z_0 is reinforced by variation in W_0. For example, $\partial E(t)/\partial T|_{Z_0}$ increases in the technology of skill production $\mu_E(x)$ ($\partial^2 E(t)/\partial T \partial \mu_E(x) > 0$), leading to prediction 3. More generally, the effect of life expectancy on skill $\partial E(t)/\partial T|_{Z_0}$ increases in any factor that increases the marginal value of skill $q_{e/a}(t)$ (see the expression for $\partial E(t)/\partial T|_{Z_0}$ in 83), such as initial wealth A_0 (see 86), initial skill E_0 (see 91), initial health H_0 (see 97), skill productivity $\mu_E(t)$ (see 105), and health productivity $\mu_H(t)$ (see 114).

Another type of complementarity between Z_0 and W_0 could arise from the third term in (116). This term increases in $\partial g(t)/\partial T|_{Z_0}$, which is positive for skill $E(t)$ and health $H(t)$ (see prediction 1). It is cumbersome to mathematically establish that $\partial^2 T/\partial Z_0 \partial W_0$ is positive. First, there are many such possible combinations and second these higher-order expressions are substantially more complicated to analyze (see, e.g., 78 and 79). But intuitively, two factors that both increase longevity could operate together, acting as complements. For example, it is plausible that the effect of initial assets on life expectancy $\partial T/\partial A_0$ is increasing in health.