Collateral, Rehypothecation, and Efficiency[*]

Charles M. Kahn[†]and Hye Jin Park[‡]

Last updated: April 25, 2016

Abstract

This paper studies rehypothecation, a practice in which financial institutions re-use or re-pledge collateral pledged by their clients for their own purposes. First, we show that rehypothecation has trade-off effects; it enhances provision of funding liquidity to the economy so that additional productive investments can be undertaken, but incurs deadweight cost by misallocating the asset among the agents when it fails. Second, we show that the intermediary's choices of rehypothecation may not achieve a socially optimal outcome. The direction of the conflict between the objectives of the intermediary and social efficiency depends on the terms of contract between the intermediary and his borrower; if the contract involves over-collateralization, there tends to be an excessive use of rehypothecation, and if the contract involves under-collateralization, there tends to be an insufficient use of rehypothecation. This result also offers an empirical prediction that if the initial borrower has the right to forbid rehypothecation, his choice will depend on the terms of contract.

Keywords: collateral, rehypothecation, repurchase agreement, contract theory, moral hazard
noindent JEL Classification: D53, D62, G21, G28

[†]Department of Finance, University of Illinois at Urbana-Champaign. E-mail: c-kahn@illinois.edu
[‡]Department of Economics, University of Illinois at Urbana-Champaign. E-mail: park354@illinois.edu

1 Introduction

Most financial contracts are in the form of promises to pay a certain amount of money or exchange assets on a later date at pre-arranged terms. But often these promises cannot be warranted themselves, and they need to be backed by an eligible asset or property, called collateral, such as Treasury bills in repo transactions and residential houses in mortgage contracts.[1] Generally, collateral in the finanancial contracts plays two crucial roles as emphasized in Mills and Reed (2012): (i) first, collateral provides a borrower with incentives to repay to avoid forfeiting it; (ii) second, collateral provides a lender with some insurance allowing him to collect some revenue by liquidating it in the event that the borrower defaults. In order that a certain asset can be used as collateral, however, it has to be sufficiently valuable especially to the borrower so that the lender can be assured that the borrower will repay the loan to get back the collateral.

Nonetheless, such assets that can be used as collateral are scarce in the economy and the cost of generating these assets are also non-negligible. In particular, as the volume of financial transactions has sharply increased over the last few decades, the demand for collateral has also been significantly increased, and economizing on the existing limited amount of collateral has become an important issue for market participants.[2]

A simple and probably the easiest way to save on collateral would be by re-using it. In most cases, collateral sits idle in the lender's account until the borrower repays the loan to get it back. Clearly, during the time that the collateral deposited in the lender's account, it ties up capital that the lender might have other profitable uses for. In that case, one way that the lender can access that capital is to make a loan by re-pledging the collateral (initially pledged by his borrower) to another party. From the view of liquidity provision, this re-using collateral is socially beneficial because it reduces the cost

---

[1] The oldest form of collateralized lending is the pawn shop that Holmström (2015) illustrates as: "The earliest documents on pawning date back to the Tang Dynasty in China (around 650 AD)... The borrower brings to the pawn shop items against which a loan is extended. The pawn shop keeps the items in custody for a relatively short (negotiable) term, say one month, during which the borrower can get back the item in return for repayment of the loan. It sounds simple, but it is a beautiful solution to a complex problem." For other insightful dicussion on the origin of collateralized lending, see Geanakoplos (1996).

[2] Krishnamurthy and Vissing-Jorgensen (2012) estimated the liquidity and safety premium on Treasuries paid by the investors on average from 1926 to 2008 was 72 basis points per year, which supports the idea that there has been a large and persistent demand for safe and liquid assets in the economy. Similarly, Greenwood, Hanson, and Stein (2012) emphasize the monetary premium embedded in short-term Treasury bills, and it have a lower yield than would be in a conventional asset-pricing literature.

of holding collateral for the lender, and ultimately it would benefit the borrower since the lender would be willing to provide more funding against the same unit of the collateral posted by the borrower. From the view of the economy as a whole, the same collateral is used to support more than one transaction, and it creates a 'collateral chain' in the system which increases interdependence among the agents.

Inarguably, rehypothecation has been one of the most popular devices for many broker-dealer banks to serve their own funding liquidity needs before the crisis. However, as reported by Singh (2010, 2011), after the failure of Lehman Brothers in 2008, rehypothecation significantly dropped as hedge funds (the clients of those investment banks) became wary of losing access to their collateral, and limited the amount of the assets that are permitted to be re-pledged.[3] At the same time, regulation on rehypothecation has also been advocated by legislators and policy-makers.[4] Nevertheless, understaning of the economics underlying this practice is still incomplete, and there are still considerable debates on how to regulate rehypothecation as being made clear by the asymmetry of the rules on rehypothecation across different nations.[5]

This paper addresses some basic, but not yet completely answered questions about this practice of re-using collateral: under what circumstances 'rehypothecation' – the practice in which the receiver of collateral re-uses, re-pledges, or sometimes even sells the collateral to another party for its own trading or borrowing – arises; how it creates a collateral chain in the system; what benefits and costs it produces; and whether decentralized decisions made by each individual to participate in rehypothecaion achieves a socially efficient outcome.

To answer these questions, we adopt the framework of Bolton and Oehmke (2014) that is in turn based on Biais, Heider, and Hoerova (2012), in which a borrower who is subject

---

[3]Singh, in a series of papers (beginning with Singh, 2010, 2011; see also Singh 2014), emphasizes the role of collateral in the financial market, and discusses how the change in the use and re-use of collateral after the crisis affected the financial system. For example, Singh (2010, 2011) reports that in 2007, in the run up to the crisis, the value of collateral held by the largest U.S. investment banks, Lehman Brothers, Bear Stearns, Morgan Stanley, Goldman, Merrill and JPMorgan, that was permitted to be rehypothecated was around $ 4.5 trillion. Post the crisis, in 2009, the value of collateral held by the U.S, investment banks that was permitted to be rehypothecated dropped to $2 trillion, which is less than half its former size.

[4]On the regulatory side, the Dodd-Frank Act requires in most swap contracts, the collateral be held in a segragated account of a central counterparty.

[5]Under SEC rule 15c3-3, a prime broker may rehypothecate assets to the value of 140% of the client's liability to the prime broker. In the U.K., there is no limit on the amount that can be rehypothecated. See Monnet (2011) for more detailed explanation on the difference in regulatory regimes on rehypothecation across countries.

to a moral hazard problem and required to post collateral to prevent him from engaging in risk-taking actions. Not surprisingly, within this basic framework, we show that a positive NPV investment of the borrower with limited liability cannot be undertaken without posting the borrower's asset as collateral, which was already demonstrated in many previous literature on collateralized borrowing.[6] These models, however, do not consider the risk on the other side that the lender might fail to return the collateral as well as any incentives to use it for their own purposes. In contrast, our model incorporates the possibility of re-using collateral by the counterparty and the risk associated with it, thereby offering the first formal welfare analysis on rehypothecation.

Another important feature of our model is that the borrower transfers collateral to the lender at the time of the beginning of the contract. In other words, collateral in our model is like a repurchase agreement in Mills and Reed (2012): the borrower transfers his asset to the lender at the time a contract is initiated and buys it back at a later point.[7] This contrasts to most of the previous works on collateral in which collateral is transferred to the lender after final pay-offs are realized, or at the time when the default of the borrower actually occurs.

This early transfer of collateral, however, introduces another risk that the lender may not be able to return collateral at the time when the borrower wants to repurchase it.[8] Indeed, as observed from the failure of Lehman Brothers in 2008 and MF Global in 2011, this is not simply a theoretical possibility. In consideration of this, we introduce the counterparty risk – the lender might lose collateral too frequently – into the baseline framework, and we show that if the risk is too high, it makes too costly for the borrower to post its asset as collateral, and he may not want to post his asset as collateral upfront. As a result, the positive NPV project of the borrower cannot be undertaken in this case since non-collateralized borrowing is not feasible when the borrower is subject to moral hazard.

---

[6]Holmström and Tirole (1998, 2011) shows that the moral hazard problem of the borrower makes the firm's pledgeable income less than its total value, which leads to a shortage of liquidity for its investment in some states. Also, Shleifer and Vishny (1992), Bernanke, Gertler, and Gilchrist (1994), and Kiyotaki and Moore (1997) concern a firm's financing problem constrained by its net wealth.

[7]According to Mills and Reed (2012), this is relevant especially in shadow banking sectors in that the loan is short-term and has large value, which makes enforcing a transfer of collateral after bankruptcy of borrowers highly costly. In constrast, small value loans between a bank and a consumer, it is relatively easy to seize collateral from borrowers at a later point, for example, in a mortgage constract, a house as collateral is not transferred to the bank until the borrower defaults on the loan.

[8]Mills and Reed (2012) discuss the effect of this counterparty risk on the form of the optimal contract in a different context.

Building on this basic intuition in the two-player model, we extend it into the three-player model to more explicitly describe how rehypothecation introduces the risk of counterparty failure and specifies the condition under which rehypothecation is socially efficient. Our results show that the efficiency of rehypothecation is determined by the relative size of the two fundamental effects. Rehypothecation lowers the cost of holding collateral and makes the illiquid collateral more liquid, thereby providing more funding liquidity into the market. On the other hand, rehypothecation failure – the counterparty failure to return the collateral to the borrower who posted it – may incur deadweight costs in the economy.

One difficulty in this general argument is that it is not obvious by which channel the rehypothecation failure incurs deadweight costs, and this was not been clearly answered in most of the previous works on rehypothecation (we discuss further on those papers in the literature review). While there could be several channels that the rehypothecation failure incurs deadweight costs in the economy, this paper focuses on the channel that the rehypothecation failure leads to misallocation of the assets posted as collateral.

This misallocation of the assets crucially depends on the following two types of market frictions: (i) we assume that the asset is 'illiquid' in the sense that the asset is likely to be more valuable to the initial owner than to the other agents – think of the asset as an intermediate good that the initial owner uses it for its own production and he has a better skill to manage it than do the other agents in the economy; (ii) we also consider a possibility that some traders may not have access to some parts of the markets, and they can trade indirectly each other only through the intermediary (who has an access to all the markets). In the model, the asset provider and the cash lender make separate contracts with the intermediary who transfers the collateral between them. Taken together, if the intermediary fails, the asset ends up being in the wrong hands: the asset cannot be returned to the initial owner (the asset provider) who values it the most, but instead it is seized by the third party (the cash lender) to whom the collateral may not be as useful.

Finally, we ask the question whether an individual agent's decision to participate in rehypothecation achieves a socially optimal outcome. To answer this question, we endogenize each individual's participation decision about rehypothecation, and investigate whether their objectives are aligned with social efficiency. We show that in general, the ex-post objective of the intermediary (the lender of the initial borrower in the model) may conflict with what would be ex-ante efficient.

The direction of this conflict between the intermediary's objective and the socially effi-

cient choice depends on the terms of contract between the intermediary and his borrower. If the contract involves over-collateralization, in the sense that the value of collateral to the borrower exceeds the payment for recovering it, there tends to be an excessive use of rehypothecation by a holder of collateral. Intuitively, this is because when the contract involves over-collateralization, there is a negative externality of 'not' returning the collateral to the borrower, which is reflected in the spread between the borrower's private value on his collateralized asset and the payment for recovering it – from the perspective of the borrower, he is supposed to repurchase the collateral at a price lower than his valuation on it. However, the intermediary does not internalize this private cost to the borrower from failure of returning collateral, and thus he sometimes wants to participate in rehypothecation even when rehypothecation is inefficient –the social cost (which includes this private cost to the borrower from rehypothecation failure) exceeds the benefit. Similarly, if the contract involves under-collateralization, there tends to be an insufficient use of rehypothecation.

Furthermore, this result has the additional implication that if the initial borrower has the right to permit or restrict rehypothecation, his choice will depend on whether the optimal contract between him and his lender induces under- or over-collateralization: if the optimal contract invloves under-collateralization, he tends to permit rehypothecation more often; if the optimal contract involves over-collateralization, he tends to be reluctant to permit rehypothecation.

2 Related Literature

This paper relates to the literature that considers collateral as an incentive device to deal with a borrower's moral hazard problem, for example, Holmström and Tirole (1998), Biais, Heider, and Hoerova (2012), and Bolton and Oehmke (2014). Holmström and Tirole (1998) consider a borrower with limited liability who has to prepare himself against uncertain liquidity shocks tomorrow by making state contingent contracts in advance to provide liquidity in a state of liquidity shortage tomorrow. A fundamental assumption in their analysis is that there is a wedge between the value of the total income of the borrower and the pledgeable income to the lender (in their terminology, inside liquidity or collateral), and as a microfoundation to it, they prove that such a wedge can arise in the optimal contract when the borrower is subject to the moral hazard problem.

In the context of derivatives trading, Biais, Heider, and Hoerova (2012) discuss the role of collateral (margin) to mitigate the moral hazard problem of the derivatives providers. The key feature of their model is that the agent is required to post collateral 'before' that final payoffs are realized. One of the advantages of depositing collateral in the lender's account is that it can prevent the collateral from being affected by the borrower's risk taking behaviors as new information arrives after the contract begins. In a similar vein, Bolton and Oehmke (2014), based on the framework of Biais, Heider, and Hoerova (2012), analyze current previledged bankruptcy treatment of derivatives and show that the seniority of derivatives can be inefficient by transfering default risk to creditors in the debt market, even if the default risk can be born more efficiently in the derivative market.

This paper is closely related to those models in that the borrower has to transfer his asset as collateral to the lender at the time when the contract begins, and buys it back later on. However, there are substantial differences between these models and ours in the following two aspects: (i) In these models, posting collateral itself is a costly behavior as it transfers a 'liquid' capital to the lender's account which yields a relatively low return than does the borrower's account, thereby incurring deadweight costs in the economy. In contrast, this paper assumes that collateral is an 'illiquid' asset, and thus posting collateral does not incur any costs at the time when the borrower transfers collateral to the lender, but it may incur some costs if the borrower fails to repay and cannot recover it in a later period.[9] (ii) This paper also considers a default risk by the receiver of collateral which was absent in those models; they assume a central counterparty (CCP) sitting between the borrower and the lender, and collateral deposited in the CCP's margin account is ring fenced not only from the pledgor's moral hazard but also from any other credit risks of the receiver. In contrast, in our setting, rehypothecation renders the collateral open to the receiver's default risk, that is, the collateral may not be returned to the pledgor if the receiver defaults having re-pledged the collateral to another party.

This paper also relates to the literature in which collateral is in the form of a repurchase aggrement, for example, Mills (2004, 2006), Mills and Reed (2012), Oehmke (2014). In particular, Mills and Reed (2012) describe collateral as a repurchase agreement when the lender lacks an enforment technology to seize collateral in the event of default. This early transfer of collateral, however, introduces an additional incentive constraint to the lender that he may not return the collateral to the borrower. With this double lack of

---

[9]According to Mills and Reed (2012), an alternative interpretation is that costs are borne when posting collateral, but may be recovered if the borrower buys it back from the lender at a later point.

commitment, they discuss how the default risk of lenders (failure to return collateral to their borrowers) affects the optimal allocation, and they show that actual defaults by lenders will not happen at the optimum. On the other hand, in our model, the defaults by lenders may occur exogenously as long as they participate in rehypothecaion, and thus the defaults can still occur at the optimum. In addition, we assume that the lender is risk-neutral, and do not consider the insurance role of collateral as in their model.

Another interesting feature in our setting is that some traders (in practice, banks and broker-dealers) have dual positions as a lender and a borrower when they re-pledge collateral received from their borrowers. A similar concept also appears in the literature on business cycles and collateral constraints, for example, Moore (2011), Getler and Kiyotaki (2010), and Gertler, Kiyotaki, and Queralto (2011) in that banks not only plays a role as an intermediary between capital producing firms (outside borrowers) and households (outside lenders), but they also borrow and lend each other, that is, they mutually hold gross positions. In particular, Moore (2011) addresses the question of why banks hold gross positions in the current financial system and whether these mutual gross positions give rise to a systemic risk in the economy. His analysis shows that those mutual gross positions among banks help make more funds flow in the system, thereby increasing investment activities, while at the same time, they make the system more susceptible against a shock which might result in a systemic failure.

More broadly, this paper is related to the literature on the supply and demand for safe and liquid assets, such as Gorton and Pennachi (1990), Dang, Gorton, and Holmström (2013), Gorton and Ordoñez (2013). In this literature, safe assets are provided by the financial intermediary when there is a demand for such assets in the economy, possibly due to informational problems. Similarly, in this paper, rehypothecation plays a role to provide liquidity to the system by circulating the limited amount of collateral. For the empirical analysis, Krishnamurthy and Vissing-Jorgensen (2013) show the existence of a large and persistent demand for safe and liquid assets, and argue this is a key driver of the prevalence of short-term debt in the economy. Also, Gorton, Lewellen, and Metrick (2012) and Aitken and Singh (2010) discuss the role of the shadow banking system in providing safe and liquid assets.

After the financial crisis in 2007, there has been growing interest in rehypothecation from both policy groups and academic researchers. Monnet (2011) discusses the possible pros and cons of rehypothecation as well as the current debates on the regulations on rehypothecaiton. Bottazi, Luque, and Pascoa (2012) develop the equilibrium model of

repos and demonstrate that prices of securities in the repo markets increase due the leverage built up along the process of rehypothecation. Andolfatto, Martin, and Zhang (2014) analyze the effect of rehypothecation on monetary policy and argue that restrictions on rehypothecation generally improve social welfare by increasing the value of cash which could be the only accepted means of exchange in some countries. Maurin (2014), based on the general equilibrium model with collateral constraint of Geanakoplos (2010), discusses the effectiveness of rehypothecation compared with other trading techniques such as tranching and pyramiding. He shows that rehypothecation has no effect on trading outcomes in complete markets, and thus the effectiveness of rehypothecation depends on the market structure.

In the context of a repo market, Lee (2015) discusses a tradeoff of rehypothecation between economic efficiency and financial stability. She emphasizes that a sudden decline of rehypothecation can lead to an inefficient repo run by creating a positive feedback loop between the repo spread and fire-sale discounts. Eren (2014) and Infante (2014) consider the case in which a dealer bank earns (free) liquidity by using its position as an intermediary between collateral providers and cash lenders, and show that, through this rehypothecation process, the dealer bank earns additional liquidity by setting larger margins to the collateral providers than to the cash lenders. Finally, Muley (2016) distinguishes between rehypothecation and securitization, and argues that rehypothecation is more likely when the initial borrower is able to effectively monitor the behavior of the collateral holder.

The paper is organized as follows. The basic model with two players is presented in the next section. In Section 4, basic model is extended to three players to allow for rehypothecation and analyzes the welfare of equilbrium under rehypothecation. Section 5 illustrates the conflicts between the intermediary's choice of rehypothecation and the social efficiency, and Section 6 concludes.

3   A Baseline Model

In this section, we build a simple two-player model following the approach of Bolton and Oehmke (2014) in which a borrower is subject to a moral hazard problem to engage in a risky action that is not desirable from the lender's perspective.

In the model, the moral hazard problem leads to a shortage of pledgeable income of the

borrower – in the sense that the future pledgeable return of the the borrower's investment is not enough to repay the initial cost of the investment, and thus prevents the borrower from raising funds by issuing a simple debt contract.

In such a scenario, requiring the borrower to post his valuable asset as collateral can mitigate the borrower's incentive to engage in the risky action, and makes it possible for captial to be transferred from the lender to the borrower.[10] To be specific, collateral increases the borrower's pledgeable income in the following two ways: (1) it provides the borrower with an incentive not to engage in a risk-taking behavior in order to get back his collateral from the lender; (2) it provides the lender with some compensation in case that the borrower defaults by allowing the lender to seize and liquidate it.

Next, we incorporate into this basic model an additional risk that the lender might be unable to return the borrower's collateral, and show that if the risk is too high, the borrower will be reluctant post his asset as collateral at all. As a result, even when collateralized borrowing is available, borrowing will not occur.

In the following subsections, we describe a basic structure of the model and show how posting collateral helps to enhance the provision of funding liquidity to the economy in which the borrower suffers from the shortage of pledgeable income due to the moral hazard problem. Then, we show this positive role of collateral in enhancing the supply of liquidity into the system can be diminished if the counterparty risk of losing collateral is introduced. In the next section, we extend this basic two-player model to the three-player model in which the lender can rehypothecate the borrower's collateral to the third party, and examine how the basic observations found within the two-player model are carried over to the extended model and see what other implications we can derive from it.

3.1 Players and Endowments

There are two periods, date 0 and date 1, and two types of agents in the economy, a firm A and an outside investor B. All agents are risk-neutral and consume at the end of date 1. For simplicity, the price of date 1 consumption good is normalized to be 1.

---

[10]In our model, we assume that the borrower has an indivisible asset whose value is not affected by the borrower's action. In comparison, Bolton and Oehmke (2014) consider the case in which the borrower has a certain amount of assets which may yield high or low return depending on whether the borrower takes a safe or risky action, and depositing a part of these assets into the lender's margin account (i.e. posting these assets as collateral) keeps the value of these assets stable and mitigates the borrower's incentive to engage in risky action by lowering the amount of assets under his management.

At date 0, firm A has an opportunity of an investment which requires an immediate input at date 0 to produce an outcome at date 1. We assume that the outcome of the investment is uncertain and can take two values; if the investment succeeds, the investment produces $R > 1$ units of the date 1 good (measured per unit of inputs) and if it fails, it produces zero units,

$$\text{outcome of investment} = \begin{cases} R & \text{if the investment succeeds} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

These outcomes are costlessly observable to the outside investors.

However, at date 0, A is endowed with no capital that can be spent as an input for his investment, but only one unit of indivisible asset which is illiquid in the following two senses. First, it yields the consumption goods only at the end of date 1. Second, it produces more outcomes when it is in the hands of the initial owner, firm A, than in the hands of the outside investor, B – for example, a stock, bond, or security included in A's portfolio is likely to fit better for A's portfolio but not for the other's. We denote that the asset yields Z units of the good if it is held by A at the end of date 1, while it yields less than that, $Z_0(< Z)$ units of the good if it is held by B at that time.

On the other hand, the outside investor B is endowed with a large amount of capital that can be spent as an input fot A's investment. Thus, A cannot undertake the project alone, and has to borrow capital from B. For simplicity, we assume that A tries to borrow funds for his investment from B by issuing simple debt; A receives cash from B at date 0 by promising to pay a certain fraction of his investment outcome to B at date 1.

3.2 Moral Hazard and Limited Liability

Following the approach of Bolton and Oehmke (2014), we assume that the probability of the success of the borrower's investment depends on his hidden action. We assume that A can choose either a safe or a risky action, denoted by $a \in \{s, r\}$ where $a = s$ represents the safe action and $a = r$ represents the risky action. The safe action leads to a high probability of success of the investment, which we take for simplicity to be 1 and

the risky action lower probability of success, $p < 1$,

$$\text{probability of success} = \begin{cases} 1 & \text{if A takes safe action } (a = s) \\ p & \text{if A takes risky action } (a = r). \end{cases} \tag{2}$$

On the other hand, the risky action gives firm A a private benefit $b > 0$ (measured per unit of inputs).

Taken together, the (expected) average return of the investment is given by

$$\text{return of the investment} = \begin{cases} R & \text{if A takes safe action } (a = s) \\ pR + b & \text{if A takes risky action } (a = r). \end{cases} \tag{3}$$

In addition we assume that the parameters satisfy the following two assumptions.

Assumption 1. $\min\{R, pR + b\} > 1 > pR$.

The first inequality implies that A's project is efficient regardless of his action. The second inequality implies that, from the persepective of B, it is profitable only if A takes the safe action. To understand the second inequality, suppose B invests capital I into A's project. Then, if A takes the safe action, the maximum level of the expected payment by A is RI, which is greater than the investment cost I by the first inequality, and if A takes the risky action, it reduces to pRI (note that the private benefit bI cannot be pledgeable), which is smaller than the investment cost I by the second inequality.

Assumption 2. $R - 1 < p(R - 1) + b$.

This assumption implies that when A invests with the borrowed money from B, A will always find it profitable to take the risky action rather than the safe action for any given contract (I, X) such that $X \geq I$ (or, equivalently, for any contract with a positive interest rate). To see this, notice that the left hand side represents A's expected net surplus (per unit of the inputs) after paying out the investment cost to B if he takes the safe action and the right hand side represents A's expectied net surplus if he takes the risky action.

Combining assumptions 1 and 2, one can infer that A cannot borrow funds for his investment from B, because B expects that A will take the risky action after borrowing, and he will end up with negative profit. Formally, Assumption 2 implies that A will always take the risky action after borrowing, and the expected loan payment by A will

be at most pR (per unit of inputs), but this is not enough to cover the cost that B spent for A's project by Assumption 1.

In the following subsections, we build a series of examples to show how collateral helps to provide funding liquidity to A's investment in case that A cannot borrow funds for his investment from B due to the moral hazard problem.

As a benchmark, we consider the first case in which A issues uncollateralized debt. In this case, we show that A's investment cannot be undertaken if Assumption 1 and 2 hold as we described before.

Second, we consider the case in which A posts his endowed asset as collateral to back up his debt. We show that when posting collateral, A's incentive to engage in the risky behavior becomes weaker as it exposes A to the risk of losing collateral if he defaults. Therefore, collateral helps A to raise funding for his investment from B.

Lastly, we add counterparty risk to the model; the possibility that B might lose A's collateral with some positive probability. We show that when this counterparty risk is too high, A may not want to undertake the investment at the expense of losing his asset, and thus borrowing will not occur at all.

3.3   Benchmark: Uncollateralized Borrowing

Let us start by considering the contracting problem between A and B where A issues debt which is solely backed by the future return of the investment. For simplicity, we assume that A has all the bargaining power and makes a take-it-or-leave-it offer to B. We assume that B's outside option pays utility of $J \geq 0$, and B will accept the offer as long as he can receive utility greater or at least equal to J.

Timing is as follows. At date 0, A borrows the investment cost, denoted by I_a, from B, and then takes either the safe or risky action, denoted by $a \in \{s, r\}$. At date 1, the investment outcome is realized and A pays a part of the return of the investment, denoted by X_a, to B (hereafter, the subscript a stands for which type of action is taken by A).

Note that depending on A's action, the optimal contract takes either of the two forms: in one case, A takes the safe action and in the other case, A takes the risky action. Let us first consider the case in which A takes the safe action, $a = s$. In this case, the contracting problem is to choose (I_s, X_s) which solves the following maximization problem.

$$\max_{I_s, X_s} RI_s - X_s \tag{4}$$

subject to

$$RI_s - X_s \geq p(RI_s - X_s) + bI_s \qquad (IC_s)$$

$$X_s - I_s \geq J \qquad (P_s)$$

$$RI_s \geq X_s \qquad (R_s)$$

The objective function is A's expected utility when A takes the safe action. With probability 1, the investment yields the return RI_s and A has the remaining amount after paying off the loan X_s out of this to B. The incentive constraint (IC_s) implies that A's expected profit when A takes the safe action on the left hand side is greater than that when A takes the risky action on the right hand side. The participation constraint (P_s) ensures that B's expected profit from lending cannot be less than the reservation value J. Lastly, the resource constraint (R_s) says that A cannot pay more than what he has, that is, the payment is bounded above by the return from the investment when it succeeds, RI_s (note that if the investment fails, it yields zero ouput, and A does not make any payments to B).

Next, consider the case in which A takes the risky action, $a = r$. In this case, the contracting problem is to choose (I_r, X_r) which solves the following problem,

$$\max_{I_r, X_r} \; p(RI_r - X_r) + bI_r \qquad (5)$$

subject to

$$RI_r - X_r \leq p(RI_r - X_r) + bI_r \qquad (IC_r)$$

$$pX_r - I_r \geq J \qquad (P_r)$$

$$RI_r \geq X_r \qquad (R_r)$$

The objective function is A's expected utility when A takes the risky action. A obatains the return RI_r from the investment and pays off the loan X_r with probability p and also receives the private benefit bI_r from misbehavior. The incentive constraint (IC_r) implies that A's expected profit when A takes the risky action which is on the right hand side is greater than that when A takes the safe action which is on the left hand side. The participation constraint (P_r) and the resource constraint (R_r) are the same as in the previous case.

Taking the two subcases together, the optimal contract is to choose a profile of (I_a, X_a, a) where $a \in \{s, r\}$ which solves the following problem.

$$\max_{a \in \{s,r\}} \mathbf{1}_S(a)(RI_s - X_s) + (1 - \mathbf{1}_S(a))[p(RI_r - X_r) + bI_r] \tag{6}$$

where $\mathbf{1}_S(\cdot)$ is the indicator function where $S = \{s\}$ and (I_s, X_s) solves subproblem (4) and (I_r, X_r) solves subproblem (5).

However, the solution to the maximization problem above may not exist under some parameter values. In other words, it may not be feasible to finance A's project by issuing simple debt, which is solely backed by the future return of the project.

Lemma 1. *Suppose Assumption 1 and 2 hold. Then, uncollateralized debt financing for A's project is not feasible.*

The intuition behind this result goes as follows. First, note that if A takes the risky action, A's investment is not profitable for the outside investor, B. In other words, the pledgeable return of the investment is smaller than the investment cost (Assumption 1 implies that $pRI < I$ for any investment scale I), which means that B cannot recover the cost that he provided to A's project if A misbehaves. However, with limited liability, in case of default, A earns zero profit but does not have further punishment, and A might find it more profitable to take more risk when he invests with the borrowed money from B. And, it turns out that this is always ths case if Assumption 2 holds.

Therefore, combining both Assumptions implies that B knows that A will always take the risky action with the borrowed money, which will yield negative profit to B, and thus lending will not occur at all and the investment cannot be undertaken.

3.4   Collateralized Borrowing

In the previous section, we showed that A's project cannot be funded with uncollateralized debt if Assumption 1 and 2 hold. Suppose now that A is required to post his endowed asset, which is worth Z to A himself and $Z_0 < Z$ to B, as collateral. In this section, we show that in such case, posting collateral helps A's investment to be funded in the following two ways: (i) posting collateral incentivizes A to take the safe action by introducing the risk of forfeiting it if he defaults; (ii) collateral provides B with some compensation in case that A defaults by allowing B to seize it.

15

To show this formally, let us consider the contracting problem between A and B when A posts his asset as collateral. As before, A is assumed to have all the bargaining power and makes a take-it-or-leave-it offer to B. At date 0, A borrows the investment cost I_a from B and deposits his endowed asset in B's account (or, pledges it as collateral), and then takes either the safe or risky action, $a \in \{s, r\}$. At date 1, if the investment succeeds, A makes the promised payment X_a to B, or if the investment fails, A defaults and B seizes the asset posted by A.

As in the previous section, in order to solve for the optimal contract, we consider the two possible cases separately. First, we begin with the case in which A takes the safe action, $a = s$. The contracting problem between A and B in this case is to choose (I_s, X_s) to solve the following maximization problem.

$$\max_{I_s, X_s} RI_s - X_s \tag{7}$$

subject to

$$RI_s - X_s \geq p(RI_s - X_s) + bI_s - (1-p)Z \tag{IC_s'}$$

$$X_s - I_s \geq J \tag{P_s'}$$

$$RI_s \geq X_s \tag{R_s'}$$

The objective function is as before. The incentive constraint (IC_s') now has the additional term $-(1-p)Z$ on the right hand side. This captures the fact that there is now an additional loss from taking the risky action, which is calculated by the probability of default, $1-p$ times the private value of collateral to A, Z. In constrast, if A takes the safe action, A will always get back his collateral. Hence, when posting collateral, the return when taking the safe action increases relative to that when taking the risky action, thereby incentivizing A to take the safe action. The participation constraint (P_s') and the resource constraint (R_s') are the same as before.

Next, consider the case in which A takes the risky action. In this case, the contracting problem is to choose (I_r, X_r) which solves the following problem,

$$\max_{I_r, X_r} p(RI_r - X_r) + bI_r - (1-p)Z \tag{8}$$

subject to

$$RI_r - X_r \leq p(RI_r - X_r) + bI_r - (1-p)Z \qquad (IC'_r)$$

$$pX_r - I_r + (1-p)Z_0 \geq J \qquad (P'_r)$$

$$RI_r \geq X_r \qquad (R'_r)$$

The objective function is A's expected utility when A takes the risky action which is the same as before except that there is additional term $-(1-p)Z$, which captures the cost of losing collateral in case of default with probability $1-p$ when A takes the risky action. The incentive constraint (IC'_r) shows that compared to the case without collateral, the return from taking the risky action on the right hand side decreases by $(1-p)Z$ due to the loss of value from forfeiting it in case of default. The participation constraint (P'_r) has the additional term $(1-p)Z_0$, which means the compensation value that B earns from liquidating collateral, Z_0, if A defaults with probability $1-p$. Again the resource constraint (R'_r) is the same as in the previous case.

Taking these together, the optimal solution is a profile of (I_a, X_a, a) which solves the following problem.

$$\max_{a \in \{s,r\}} \mathbf{1}_S(a)(RI_s - X_s) + (1 - \mathbf{1}_S(a))[p(RI_r - X_r) + bI_r - (1-p)Z] \qquad (9)$$

where $\mathbf{1}_S(\cdot)$ is the indicator function where $S = \{s\}$ and (I_s, X_s) solves subproblem (7) and (I_r, X_r) solves subproblem (8).

3.4.1 Optimal Solution

Our next result shows that if B's outside utility J is sufficiently small, there exists a solution to the problem described above, and to characterize it.

Proposition 1 (Optimal Contract under Collateralized Borrowing). *Suppose Assumption 1 and 2 hold. If $J \leq \min\left\{\frac{1-p}{b}(R-1)Z, \frac{1-p}{b}\left(pR + b(Z_0/Z) - 1\right)Z\right\}$, there exists an optimal solution to problem (9). In this solution*

$$(I_a, X_a; a) = \begin{cases} \left(\frac{Z-J}{1-\mathcal{B}}, \frac{Z-\mathcal{B}J}{1-\mathcal{B}}; s\right) & if \quad U_s \geq U_r \\ \left(\frac{(1-p)Z_0-J}{1-pR}, R\frac{(1-p)Z_0-J}{1-pR}; r\right) & if \quad U_s < U_r \end{cases} \qquad (10)$$

where $\mathcal{B} \equiv R - \frac{b}{1-p}$, $U_s \equiv RI_s - X_s$, and $U_r \equiv (pR + b)I_r - pX_r - (1-p)Z$.

Depending on parameter values, either the safe or the risky action can arise in the optimal contract. In either case, the participation constraint is binding at the optimum; player B receives exactly J in value.

In the subcase where the risky action is optimal $(a = r)$ the resource constraint is also binding; thus I_r and X_r are defined by the two equalities:

$$pX_r + (1-p)Z_0 = J + I_r \tag{11}$$

$$RI_r = X_r \tag{12}$$

In other words, when the investment is successful the entirety of the payout is given to B. Since this is not enough alone to compensate for the initial investment by B, the remnant of the compensation comes from the value of the collateral to B; the more valuable the collateral, the larger the initial investment.

Roughly speaking, the parameter J measures the profitability of B's lending activity. If lending is sufficiently competitive (J close to 0), the investment in the risky subcase tends to be undercollateralized; at least relative to B's valuation, collateral is less than the required repayment, $Z_0 < X_r$.

In the subcase where the safe action is optimal repayment cannot be pushed to the limit of the resource constraint, for if A were forced to pay out the full amount of the proceeds of the investment, he would not be willing to take the safe action. Instead the incentive constraint binds first, and so I_s and X_s are defined by the two equalities:

$$RI_s - X_s = p(RI_s - X_s) + bI_s - (1-p)Z \tag{13}$$

$$X_s - I_s = J \tag{14}$$

As before, increases in the value of the collateral relax the constraints on the problem and increase the amount of investment. Here, however, the relevant value is the value to the borrower, not the lender, because the collateral is being used as an incentive, not a repayment. Again roughly speaking, the need to maintain the incentives for safe behavior increases the collateral needed to back the borrowing. As J approaches 0, whether this extra consideration is sufficient to lead to overcollateralization depends on the sign of the

quantity $\mathcal{B}$; if it is negative, then $X_s < Z$.

The effect of parameter values on the choice between the safe and risky subcases can be analyzed by using the results of the proposition. For example in the case where $J = 0$ and $p = 0$, the formulas for A's utility under the two subcases reduce to

$$U_s = \frac{(R-1)Z}{1+b-R}, \quad U_r = bZ_0 - Z.$$

The risky action becomes relatively more attractive as the private benefit b increases and B's evaluation of collateral, Z_0, increases. The safe action becomes relatively more attractive as its return increases and as the value to A of retaining the collateral increases.

3.5   Counterparty Risk under Collateralized Borrowing

Next, we consider the possibility that B might lose A's collateral. Suppose this happens with a positive probability $1 - \theta \in (0, 1)$, and if B loses the collateral, he is not eligible to receive the payment from A – for example, think of the collateral as an evidence that B has a claim on A's debt. As in the previous section, we can characterize the optimal contract in this case by solving the two sub-problems, for the case in which A takes the safe action and the other case in which A takes the risky action, and then choose either of them which yields a higher utility for A.

Formally, the optimal solution in this case is a profile (I_a, X_a, a) where $a \in \{s, r\}$ which solves the following problem.

$$\max_{a \in \{s,r\}} \mathbf{1}_S(a)[RI_s - \theta X_s - (1-\theta)Z] + (1 - \mathbf{1}_S(a))[p(RI_r - \theta X_r) + bI_r - (1 - p\theta)Z] \quad (15)$$

where $\mathbf{1}_S(\cdot)$ is the indicator function where $S = \{s\}$ and (I_s, X_s) solves the following subproblem,

$$\max_{I_s, X_s} RI_s - \theta X_s - (1-\theta)Z \quad (16)$$

subject to

$$RI_s - \theta X_s - (1-\theta)Z \geq p(RI_s - \theta X_s) + bI_s - (1 - p\theta)Z \quad (IC_s'')$$

$$\theta X_s - I_s \geq J \quad (P_s)$$

$$RI_s \geq X_s \quad (R_s)$$

19

and (I_r, X_r) solves the following subproblem,

$$\max_{I_r, X_r} p(RI_r - \theta X_r) + bI_r - (1 - p\theta)Z \tag{17}$$

subject to

$$RI_r - \theta X_r - (1 - \theta)Z \le p(RI_r - \theta X_r) + bI_r - (1 - p\theta)Z \tag{IC_r''}$$

$$\theta[pX_r + (1 - p)Z_0] - I_r \ge J \tag{P_r}$$

$$RI_r \ge X_r. \tag{R_r}$$

The objective function in the first subproblem (in which A takes the safe action) shows that compared to the case without the counterparty risk, the expected loan payment decreases from X_s to θX_s. This is because A repays only when B returns the collateral which occurs with probability $1 - p$. In addition, posting collateral incurs a cost $(1 - \theta)Z$ for A, which reflects the loss of the private value Z for A in case that B loses A;s colalteral with probability $1 - \theta$. Similarly, the objective function in the second subproblem (in which A takes the risky action) shows that compared to the case without the counterparty risk, the expected loan payment decreases from pX_r to $p\theta X_r$ and the cost of posting collateral increases from $(1 - p)Z$ to $(1 - p\theta)Z$, since even if A's investment succeeds, A still cannot get back collateral from B if B default, and the probability of this event is $p(1 - \theta)$. Likewise, the participation constraint (P_s) and (P_r) show that the expected revenue from lending is scaled down by θ since B cannot receive the payment from A or any compensation by seizing collateral in case that he loses it. Lastly, the incentive constraint (IC_s'') and (IC_r'') mean the same as before.

If the counterparty risk is sufficiently high, in other words, if B loses collateral too frequently, A is reluctant to post his asset as collateral due to the concern of not getting back his valuable asset, and again, A's investment will not be undertaken even when collateralized borrowing is feasible.

For example, suppose $Z_0 = 0$ and $J = 0$. Then, provided that there exists a solution to the problem that we described above, the optimal contract should induce A to take the safe action, i.e, $a = s$, and is characterized by (I_s, X_s) as follows.

$$I_s = \frac{\theta Z}{1 - \mathcal{B}}, \quad X_s = \frac{Z}{1 - \mathcal{B}} \tag{18}$$

20

where $\mathcal{B} = R - \frac{b}{1-p}$ which is smaller than 1 by Assumption 2.

Plugging this result into A's utility function, we have

$$RI_s - \theta X_s - (1-\theta)Z = \frac{[\theta R + (1-\theta)\mathcal{B} - 1]Z}{1 - \mathcal{B}}. \tag{19}$$

Then, one can notice that as θ decreases, A's utility decreases. For example, if the counterparty risk of losing collateral is sufficiently high, $\theta \to 0$, A's utility when issuing collateralized debt becomes negative,

$$\lim_{\theta \to 0} \frac{[\theta R + (1-\theta)\mathcal{B} - 1]Z}{1 - \mathcal{B}} = -Z < 0. \tag{20}$$

In such case, A will find it more profitable to keep his asset rather than making an investment by posting it as collateral. In other words, if θ is low, the expected loss of losing collateral, reflected in the terms $(1-\theta)Z$, might exceed the expected profit from the investment made by posting his asset as collateral, reflected in the terms $RI_s - \theta X_s = \theta \frac{R-\mathcal{B}}{1-\mathcal{B}} Z$, and it would be more profitable for A to just keep his asset safe and not to invest at all.

4 Collateral Chain: Rehypothecation Model

In this section, we extend the previous two-player model into a three-player model in which B repledges A's collateral to the third party, C. This extended model describes how the same single unit of collateral can be 'rehypothecated' to support multiple transactions.

As a preliminary result, using this model, we confirm that rehypothecation generates a trade-off. On the one hand, hypothecation improves the provision of funding liquidity into the economy so that additional productive investments can be undertaken. To be specific, in our model, the intermediary B can raise funds for his another productive investment by repledging A's collateral to the third party C, even in the case in which B's pledgeable income is not enough to back up the debt that B issues for funding his investment.

On the other hand, rehypothecation may incur deadweight cost when it fails, that is, when B (the intermediary who sits in the middle of the collateral chain) goes bankrupt having repledged A's collateral to the third party, C. In that case, the collateral will be seized by C and cannot be returned to the initial owner, A, who puts a higher value on

the asset than does C, leading to the misallocation of the asset.[11]

In our model, this misallocation of the asset originates from the following two frictions in the market. The first friction is a wedge between the agents' valuations on the posted asset. In particular, we assume that the initial owner of the asset evaluates his asset higher than do the other agents in the economy.[12] The second friction is a trade friction that the initial collateral owner, A, and the final cash lender, C, cannot trade on their own, and they can trade only through the intermediary, B. Therefore, in case of rehypothecation failure, these two asssumptions together lead to the misallocation of A's asset to C.

To facilitate analysis, throughout this section, we assume that B's ex-post decision whether to rehypothecate or not is given. Under this assumption, we then separately calculate the total surplus in the case where B rehypothecates and in the case where B does not rehypothecate, and compare the welfare in these two cases to examine what benefits and costs rehypothecation produces in the economy.

Next, we relax the assumption that B's decision to rehypothecate is exogenously given, and see whether his decision aligns or conflicts with the ex-ante socially efficient decision. In other words, we ask whether B does not want to rehypothecate ex post when it is socially efficient to rehypothecate, or B prefers to rehypothecate ex post when it is inefficient to rehypothecate.

4.1 A Model

4.1.1 Players and Endowments

There are three periods, date 0, 1 and 2, and three players, A, B, and C. We assume that all the agents are risk-neutral and consume only at the end of date 2. For simplicity, we take the price of date 2 consumption good to be 1.

At date 0, A has an opportuntiy of an investment which has the same feature as in the previous baseline model, except that it produces the outcome after two periods, at date 2, not date 1. Also, we assume that at date 0, A is endowed with a single unit of indivisible illiquid asset, which is again the same as in the previous model, except that

___

[11]In general, C might be able to sell the asset and A may repurchase it from the variety of sources, but this would be a natural first step to consider. As long as there are some trading frictions between A and C, for example, if there exist non-zero transaction costs between them, there will be some deadweight loss when B fails.

[12]This is similar to the previous two player model in which collateral is more productive in the hands of the initial owner, A, than in the hands of the investors, B.

it yields the goods at the end of date 2. On the other hand, B is endowed with a large amount of capital which can be used as an input for A's investment at date 0.

At date 1, B has an investment opportunity which requires an immediate input at that time to produce an outcome at date 2, but B does not have capital that can be spent as an input for his project nor any other pledgeable assets.[13] On the other hand, C does not have access to B's investment, but has a large amount of capital that can be spent as an input for B's investment. Thus, in order that B's investment is undertaken, capital must be transferred from C to B. As in the previous section, we assume that B issues simple debt to borrow funds for his investment; B borrows funds from C at date 1 by promising to pay a part of his investment outcome to C at date 2.

Let us describe the characteristics of B's investment in more detail. First, it produces a positive outcome, Y (measured per unit of the investment cost) if it succeeds with probability θ, or zero if it fails with probability $1 - \theta$,

$$
\text{outcome of B's investment} = \begin{cases} Y & \text{with prob. } \theta \\ 0 & \text{with prob. } 1 - \theta \end{cases} \tag{21}
$$

In addition, B's investment is productive in the sense that the expected return of B's investment is greater than the investment cost (both are measured per unit of inputs).

Assumption 3. $\theta Y > 1$.

However, the outcome of B's investment is *not* verifiable to its creditor, C. For example, even when the project succeeds, B can falsely report that his investment fails, and can avoid paying the loan to C. This implies that debt financing solely backed by the future return of the investment is not feasible for B.[14]

4.1.2 Sequential Contracts: Collateral Chain

Suppose B is now allowed to repledge A's collateral deposited in his account to borrow funds from C, in other words, B can rehypothecate A's collateral. This can help to raise funds for B's investment, which otherwise cannot be funded on its own. By posting A's

---

[13]Or, we can assume that B's endowment at date 0 cannot be storable until the next period when he wants to use it as an input for his investment, for example, B faces a liquidity mismatch problem.

[14]In general, we may assume that some of the future return of B's investment can be pledgeable, but as long as it is not fully pledgeable, B cannot raise enough funding for the investment soley backed by its future returns.

collateral, B can assure C that he will make the payment to recover the collateral, so that he can receive the payment by returning it to A – note that in effect the debt between A and B is also transferred to C when the collateral is transferred from B to C.[15] In addition, collateral provides some compensation to C even by allowing C to seize the collateral in case that B defaults.

Formally, under rehypothecation, the same single unit of collateral is used to support more than one transaction, the contract between A and B at date 0 and that between B and C at date 1, thereby creating collateral chain in the economy. In this three period model, these two contracts arise sequentially, and the timing of the model proceeds as follows (also see Figure 1).

- At date 0, A borrows funds for his investment, denoted by $I_a^\dagger$, from B by pledging his asset, which is worth Z to himself and $Z_0(< Z)$ to the others, and promises to pay a part of the investment outcome, denoted by $X_a^\dagger$, to B, conditional B's returning collateral to A. After entering the contract, A then takes an action, either safe or risky, $a \in \{s, r\}$.

- At date 1, B has an investment opportuntiy, and decides whether to repledge A's collateral and undertake the investment, or keep A's collateral to return it safely to A in the next period. If B decides to rehypothecate, B borrows funds for his investment, denoted by $I_a^\ddagger$, from C by repledging A's collateral, and promises to pay a part of the investment outcome, denoted by $X_a^\ddagger$, to C for recovering A's collateral from C.

- At date 2, both A's and B's, investmet outcomes are realized and the contracts are executed according to the prearranged terms. If B pays $X_a^\ddagger$ he repurchases A's collateral from C, and then returns it to A to receive $X_a^\dagger$. However, if B defaults, C seizes A's collateral repledged by B, and it cannot be returned to A, while at the same time, A is also exempt from paying the loan $X_a^\dagger$ to B.

---

[15]For the transferability of debt, see Kahn and Roberds (2007) and Donaldson and Micheler (2015). As the conditions for debt to be transferable, Kahn and Roberds (2007) assume that the debtor and an entity who holds the debt claim can meet at some point in the future and the enforceability of debts does not diminish when it is transferred between agents. In our model, neither of these two conditions hold, and debt can be transferrable only when the collateral supporting it is transferred together.
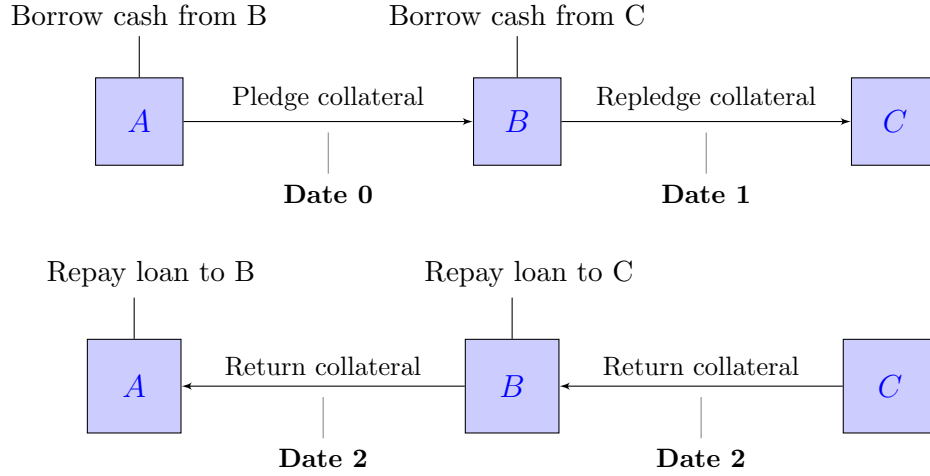
Figure 1: Timing of Contracts

4.2 Solving the Model

Next, we solve for the optimal contract in this model. Throughout this section, we focus on the case in which B's decision whether to rehypothecate A's collateral at date 1 is exogenously given – we will endogenize this into the model later on. First, consider the case in which B does not rehypothecate A's collateral at date 1. In that case, the model involves only one contract made between A and B, and effectively, it boils down to the previous two-player model.

Next, consider the case in which B rehypothecates at date 1. In that case, the model has a sequence of two contracts, the date 0 contract between A and B and the date 1 contract between B and C. Formally, the date 0 contract between A and B is defined as a profile $(I_a^\dagger, X_a^\dagger; a)$ where $I_a^\dagger$ represents the investment cost that A borrows from B by pledging his asset as collateral at date 0, $X_a^\dagger$ represents the payment promised by A to pay for getting back his asset from B at date 2, and $a \in \{s, r\}$ denotes the type of action taken by A. Similarly, the date 1 contract between B and C is defined by a profile $(I_a^\ddagger, X_a^\ddagger)$ where $I_a^\ddagger$ represents the investment cost that B borrows from C by repledging A's collateral at date 1 and $X_a^\ddagger$ represents the payment promised by B to repurchase A's collateral from C at date 2, and similarly as above, these also depend on the type of action taken by A, $a \in \{s, r\}$.

To solve for the optimal solution to this problem, we use backward induction; first, we solve for the contracting problem between B and C at date 1 by taking the date 0

25

contract between A and B as given, and then solve for the contracting problem between A and B at date 0.

4.2.1 Contracting Problem between B and C at Date 1

Let us consider the contracting problem between B and C at date 1. We assume that B has all the bargaining power and makes a take-it-or-leave-it offer to C, and if C rejects the offer, C receives reservation value 0. In the case in which A takes the safe action, $a = s$, the contracting problem between B and C is to choose $(I_s^\ddagger, X_s^\ddagger)$ which solves the following maximization problem.

$$\max_{I_s^\ddagger, X_s^\ddagger} \quad \theta(Y I_s^\ddagger - X_s^\ddagger + X_s^\dagger) \tag{22}$$

subject to

$$I_s^\ddagger \leq \theta X_s^\ddagger + (1 - \theta) Z_0 \tag{P_c}$$

$$X_s^\ddagger \leq X_s^\dagger \tag{R}$$

The objective function is B's expected utility when he makes the investment with cash borrowed by repledging A's collateral. With probability θ, the investment returns $Y I_s^\ddagger$, and B pay a part of the return $X_s^\ddagger$ to repurchase A's collateral from C, and then B delivers this collateral to receive $X_s^\dagger$ from A. The participation constraint (P_c) states that C's utility must be at least the reservation value, 0. The right hand side of (P_c) is the expected revenue from lending; C receives $X_s^\ddagger$ from B with probability θ and Z_0 by seizing the collateral if B defaults with probability $1 - \theta$. The left hand side of (P_c) is the cost $I_s^\ddagger$ that C provides to B's investment. The last constraint (R) is the resource constraint which implies that B's promise, $X_s^\ddagger$, cannot be greater than what B is going to earn by recovering the collateral from C, the expected payment that B would receive by returning the collateral to A, $X_s^\dagger$. If this does not hold, B will find it more profitable not to recover the collateral from C.

In the case in which A takes the risky action, the contracting problem is to choose $(I_r'^\ddagger, X_r^\ddagger)$ which solves the following problem.

$$\max_{I_r^\ddagger, X_r^\ddagger} \quad \theta(Y I_r^\ddagger - X_r^\ddagger + p X_r^\dagger + (1 - p) Z_0) \tag{23}$$

subject to

$$I_r^\ddagger \leq \theta X_r^\ddagger + (1-\theta)Z_0 \tag{P_c'}$$

$$X_r^\ddagger \leq pX_r^\dagger + (1-p)Z_0 \tag{R'}$$

The objective function is B's expected profit. With probability θ, B's investment returns $\theta Y I_r^\ddagger$, and B repurchases A's collateral from C at a prearranged price $X_r^\ddagger$, and then B receives $X_r^\dagger$ from A in exchange for the collateral if A's investment succeeds with probability p, or seizes the collateral (which is worth Z_0 to B) if A defaults with probability $1-p$. The participation constraint (P_c') is the same as (P_c), which implies that C's utility must be at least the reservation value, 0. Lastly, the constraint (R') implies that B's promise, $X_r^\ddagger$, cannot be greater what he is going to earn by recovering the collateral from C; B receives $X_s^\dagger$ from A in exchange for the collateral with probability p and Z_0 by seizing it if A defaults with probability $1-p$.

Then, Assumption 3 and the linearity of the problem ensure that in both cases, all the constraints are binding at the optimum, and by solving these equations simultaneously, we can write the optimal contract at date 1 as a function of the date 0 contract.

Lemma 2. *Suppose Assumption 3 holds and the date 0 contract between A and B is given. Then, the optimal contract between B and C at date 1 can be written as a function of the date 0 contract between A and B, $(I_s^\dagger, X_s^\dagger)$ if A takes the safe action and $(I_r^\dagger, X_r^\dagger)$ if A takes the risky action.*

$$(I_a^\ddagger, X_a^\ddagger) = \begin{cases} (\theta X_s^\dagger + (1-\theta)Z_0, \ X_s^\dagger) & \text{if} \quad A \text{ takes the safe action } (a=s) \\ (p\theta X_r^\dagger + (1-p\theta)Z_0, \ pX_r^\dagger + (1-p)Z_0) & \text{if} \quad A \text{ takes the risky action } (a=r) \end{cases} \tag{24}$$

In other words, B passes the collateral along to C. If neither A nor B fails, A's payment $X^\dagger$ is passed along to C. Otherwise C retains the collateral valued at Z_0. The weighted average of these two quantities is the amount that C lends to B.

4.2.2 Contracting Problem between A and B at date 0

Moving backward, we consider the contracting problem between A and B at date 0. First, we consider the case in which A takes the safe action. To facilitate analysis, we

plug the results in Lemma 2 into the objective function of the date 1 problem, so that B's expected utility can be written as a function of $(I_s^\dagger, X_s^\dagger)$ as follows.

$$\theta(YI_s^\ddagger - X_s^\ddagger + X_s^\dagger) - I_s^\dagger = \theta Y(\theta X_s^\dagger + (1-\theta)Z_0) - I_s^\dagger. \tag{25}$$

Again, we assume that A has all the bargaining power and makes a take-it-or-leave-it offer to B, and if B rejects the offer, he receives reservation value, $J > 0$. In this case, the contracting problem is to choose $(I_s^\dagger, X_s^\dagger)$ which solves the following problem.

$$\max_{I_s^\dagger, X_s^\dagger} \quad RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z \tag{26}$$

subject to

$$RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z \geq p(RI_s^\dagger - \theta X_s^\dagger) + bI_s^\dagger - (1-p\theta)Z \qquad (IC_s'')$$

$$\theta Y[\theta X_s^\dagger + (1-\theta)Z_0] - I_s^\dagger \geq J \qquad (P_B)$$

$$RI_s^\dagger \geq X_s^\dagger \qquad (R_s)$$

The objective function is A's expected utility when A takes the safe action where $1-\theta$ is the probability of default of B. The incentive constraint (IC_s'') implies that A prefers to take the safe action than the risky action, which is the same as in the previous two-player model when the counterparty risk is $1-\theta$. The participation constraint (P_B) implies that B's utility (in case that he is suppose to rehypothecate A's collateral at date 1) must be at least his outside utility, J. Lastly, the resource constraint (R_s) implies that A cannot promise pay more than what he has, i.e., the return from the investment, $RI_s^\dagger$, since he has no other source of income.

Next, consider the case in which A takes the risky action. Again, we write B's utility as a function of the date 0 contract, $(I_r^\dagger, X_r^\dagger)$ by plugging the results in Lemma 2 into the objective function of the date 1 problem described above.

$$\theta(YI_r^\ddagger - X_r^\ddagger + pX_r^\dagger + (1-p)Z_0) - I_r^\dagger = \theta Y[p\theta X_r^\dagger + (1-p\theta)Z_0] - I_r^\dagger. \tag{27}$$

Then, the contracting problem between A and B at date 0 is to choose $(I_r^\dagger, X_r^\dagger)$ which solves the following problem.

$$\max_{I_r^\dagger, X_r^\dagger} \quad (pR+b)I_r^\dagger - p\theta X_r^\dagger - (1-p\theta)Z \tag{28}$$

subject to

$$RI_r^\dagger - \theta X_r^\dagger - (1-\theta)Z \le p(RI_r^\dagger - \theta X_r^\dagger) + bI_r^\dagger - (1-p\theta)Z \tag{IC_r''}$$

$$\theta Y[p\theta X_r^\dagger + (1-p\theta)Z_0] - I_r^\dagger \ge J \tag{P_B'}$$

$$RI_r^\dagger \ge X_r^\dagger \tag{R_r}$$

The objective function is A's expected utility when A takes the risky action where p is the probability of the success of A's investment and $1-\theta$ is the probability that B loses A's collateral. The incentive constraint (IC_r'') is the reverse of (IC_s''), which implies that A prefers to take the risky action than the safe action. The participation constraint (P_B') and the resource constraint (R_r) are the same as before.

Taken together, the optimal date 0 contract between A and B is a profile $(I_a^\dagger, X_a^\dagger; a)$ which solves the following problem.

$$\max_{a \in \{s,r\}} \mathbf{1}_S(a)[RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z] + (1-\mathbf{1}_S(a))[(pR+b)I_r^\dagger - p\theta X_r^\dagger - (1-p\theta)Z] \tag{29}$$

where $S = \{s\}$ and $(I_s^\dagger, X_s^\dagger)$ solves subproblem (26) and $(I_r^\dagger, X_r^\dagger)$ solves subproblem (28).

To solve for this problem, we make the following two parametric assumptions on θ and Y to maintain consistency between this extended model with rehypothecation and the previous baseline model without rehypothecation.

Assumption 4. $\max\{\theta^2 Y R, \theta Y(pR+b)\} > 1 > p\theta^2 Y R.$

Assumption 4 is analogous to Assumption 1 in the baseline model, which implies that A's project is productive whichever action A takes, but from the perspective of B, providing funds for A's project is profitable only if A takes the safe action. To get an intuition, suppose A's collateral repledged by B is worthless to B and C, that is, $Z_0 = 0$. In that case, the maximum level of debt trasferred by repledging A's collateral is θR (per unit of inputs) if A takes the safe action, and $p\theta R$ (per unit of inputs) when A takes the risky action – note that these are multiplied by θ because A does not pay the loan if B cannot return the collateral in case of default with probability $1-\theta$. Thus, if B repledges A's collateral and undertake the investment at date 1, B's expected final

revenue is $\theta R \times \theta Y$ (per unit of inputs) if A takes the safe action, which is greater than 1 by the first inequality, or $p\theta R \times \theta Y$ (per unit of inputs) if A takes the risky action, which is smaller than 1 by the second inequality. In other words, B can receive a positive profit only if A takes the safe action.

Assumption 5. $R - \theta R < p(R - \theta R) + b$.

Assumption 5 is identical to Assumption 2, except that the number 1 on the both sides of Assumption 2 is replaced with θR. As noted before, θR is the maximum level of A's payment (per unit of inputs) if A takes the safe action [16] and B rehypothecates A's collateral. Thus, Assumption 5 can be interpreted by considering that if the value of the asset to A were to approach zero, [17] he would want to take the risky action rather than the safe action whenever he invests with the borrowed money.

Then, based on Assumption 3. 4, and 5, we show that if B's outside utility, J is sufficiently small, there exists a solution to the problem described above, and takes the following form.

Lemma 3. *Suppose Assumption 3, 4, and 5 hold. If* $J \leq \min\Big\{(1-\theta)\theta Y Z_0 + \frac{\theta(\theta^2 Y R - 1)}{\theta R - \mathcal{B}}Z, (1 - p\theta)Z_0 - \frac{\theta(1 - p\theta^2 Y R)}{\theta R - \mathcal{B}}Z\Big\}$, *there exists an optimal solution to problem (29),* $(I_a^\dagger, X_a^\dagger; a)$ *such that*

$$
(I_a^\dagger, X_a^\dagger; a) = \begin{cases} \left(\frac{\theta Y[\theta Z + (1-\theta)Z_0] - J}{1 - \mathcal{B}\theta Y}, \frac{Z + (1-\theta)\mathcal{B}Y Z_0 - (\theta/\mathcal{B})J}{1 - \mathcal{B}\theta Y}; s\right) & \text{if} \quad U_s^\dagger \geq U_r^\dagger \\ \left(\frac{(1-p\theta)\theta Y Z_0 - J}{1 - p\theta^2 Y R}, \frac{(1-p\theta)\theta Y R Z_0 - RJ}{1 - p\theta^2 Y R}; r\right) & \text{if} \quad U_s^\dagger < U_r^\dagger \end{cases} \tag{30}
$$

where $\mathcal{B} \equiv R - \frac{b}{1-p}$, $U_s^\dagger \equiv R I_s^\dagger - \theta X_s^\dagger - (1-\theta)Z$, *and* $U_r^\dagger \equiv (pR + b)I_r^\dagger - p\theta X_r^\dagger - (1-p\theta)Z$.

Finally, taken the results obtained so far together, the optimal contract in this rehypothecation model can be characterized as follows.

Proposition 2. *Suppose Assumption 3, 4, 5 hold, and* $J \leq \min\Big\{(1-\theta)\theta Y Z_0 + \frac{\theta(\theta^2 Y R - 1)}{\theta R - \mathcal{B}}Z, (1 - p\theta)Z_0 - \frac{\theta(1 - p\theta^2 Y R)}{\theta R - \mathcal{B}}Z\Big\}$. *The optimal contract under rehypothecation can be denoted by a profile* $(I_a^\dagger, X_a^\dagger, I_a^\ddagger, X_a^\ddagger; a)$ *where* $a \in \{s, r\}$ *such that*

$$
(I_a^\dagger, X_a^\dagger, I_a^\ddagger, X_a^\ddagger; a) = \begin{cases} (I_s^\dagger, X_s^\dagger, I_s^\ddagger, X_s^\ddagger; s) & \text{if} \quad U_s^\dagger \geq U_r^\dagger \\ (I_r^\dagger, X_r^\dagger, I_r^\ddagger, X_r^\ddagger; r) & \text{if} \quad U_s^\dagger < U_r^\dagger \end{cases} \tag{31}
$$

---

[16]Note that Assumption 5 is nested in Assumption 2 if and only if $\theta R > 1$.

[17]For example, think of the collateral posted by A as the letter of claim of the debt, which B must hold in order to be able to collect the debt from A, but the letter itself is unlikely to generate other private values for A.

where

$$(I_s^\dagger, X_s^\dagger) = \Big(\frac{\theta Y[\theta Z + (1-\theta)Z_0] - J}{1 - \mathcal{B}\theta Y}, \frac{Z + (1-\theta)\mathcal{B}YZ_0 - (\theta/\mathcal{B})J}{1 - \mathcal{B}\theta Y}\Big),$$

$$(I_r^\dagger, X_r^\dagger) = \Big(\frac{(1-p\theta)\theta YZ_0 - J}{1 - p\theta^2 YR}, \frac{(1-p\theta)\theta YRZ_0 - RJ}{1 - p\theta^2 YR}\Big),$$

$$(I_s^\ddagger, X_s^\ddagger) = (\theta X_s^\dagger + (1-\theta)Z_0, X_s^\dagger),$$

$$(I_r^\ddagger, X_r^\ddagger) = (p\theta X_r^\dagger + (1-p\theta)Z_0, pX_r^\dagger + (1-p)Z_0),$$

$\mathcal{B} \equiv R - \frac{b}{1-p}$, $U_s^\dagger \equiv RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z$, *and* $U_r^\dagger \equiv (pR + b)I_r^\dagger - p\theta X_r^\dagger - (1-p\theta)Z$.

4.3   Welfare Analysis: Trade-off Effects of Rehypothecation

In this section, we evaluate the social welfare under non-rehypothecation and under rehypothecation, and compare them to examine what benefits and costs rehypothecation generates in the economy. We illustrate the components of the trade-off: On the one side, rehypothecation helps provide more funding liquidity to the economy so that additional productive investment can be undertaken, by enabling the agent to use the limited amount of collateral to support multiple transactions. On the other side, it introduces an additional risk such that the intermediary might default having repledged his borrower's collateral to the third party, which is often called 'rehypothecation failure.'

A cost associated with this failure of the intermediary is that it may incur the deadweight cost of misallocating the asset, which arises in the presence of illiquidity of the asset and trading frictions; for example, the initial owner of the asset is likely to put a higher value on it than the other agents in the market, and if the intemediary repledges the initial owner's collateral to the third party, it is likely to be the case that this third party and the initial owner are indirectly connected through the intermediary, and they cannot trade on their own. This implies that, if the intermediary fails, or equivalently, rehypothecation fails, the asset remains with the third party to whom it is less valuable than to its initial owner, thereby leading to the dislocation of the asset.

Formally, if B rehypothecates A's collateral, the social welfare, denoted by W_R, can be represented by the sum of A's, B's and C's utility, respectively. First, in the case that A takes the safe action, the social welfare is given by

$$W_R(a = s) = [RI_s^\dagger - \theta X_s^\dagger + \theta Z - Z] + [\theta \bar{Y} I_s^\ddagger - \theta X_s^\ddagger + \theta X_s^\dagger - I_s^\dagger] + [\theta X_s^\ddagger + (1-\theta)Z_0 - I_s^\ddagger]. \tag{32}$$

where $(I_a^\dagger, X_a^\dagger, I_a^\ddagger, X_a^\ddagger)_{a\in\{s,r\}}$ are from Proposition 2.

Similarly, in the case that A takes the risky action, the social welfare is given by

$$W_R(a = r) = [(pR+b)I_r^\dagger - p\theta X_r^\dagger + p\theta Z - Z] + [\theta Y I_r^\ddagger - \theta X_r^\ddagger + p\theta X_r^\dagger + (1-p)\theta Z_0 - I_r^\dagger] + [\theta X_r^\ddagger + (1-\theta)Z_0 - I_r^\ddagger].$$
(33)

Simplifying, the social welfare under rehypothecation takes the following form.

$$W_R \equiv \begin{cases} (R-1)I_s^\dagger + (\theta Y - 1)I_s^\ddagger - (1-\theta)(Z - Z_0) & \text{if} \quad a = s \\ (pR + b - 1)I_r^\dagger + (\theta Y - 1)I_r^\ddagger - (1-p\theta)(Z - Z_0) & \text{if} \quad a = r \end{cases} \quad (34)$$

In other words, the social welfare under rehypothecation consists of three components: (i) the surplus generated from A's investment, which is captured by the terms $(R-1)I_s^\dagger$ and $(pR+b)I_r^\dagger$; (ii) the surplus generated from B's investment, which is captured by the terms $(\theta Y - 1)I_s^\ddagger$ and $(\theta Y - 1)I_r^\ddagger$; and (iii) the cost generated from the misallocation of the asset in case of rehypothecation failure, which is captured by the terms $(1-\theta)(Z - Z_0)$ and $(1-p\theta)(Z - Z_0)$.

On the other hand, if rehypothecation is not allowed, the social welfare, denoted by W_0, consists of A's utility and B's utility only, since further transactions between B and C cannot happen. Using the same approach as above, we can show that the social welfare under non-rehypothecation takes the following form.

$$W_0 \equiv \begin{cases} (R-1)I_s & \text{if} \quad a = s \\ (pR + b - 1)I_r - (1-p)(Z - Z_0) & \text{if} \quad a = r \end{cases} \quad (35)$$

where $(I_a, X_a)_{a\in\{s,r\}}$ are from Proposition 1. Note that the surplus is now generated only from A's investment, since B's investment cannot be undertaken. Also, provided that A's choice of action is not changed whether B rehypothecates or not,[18] rehypothecation tends to increase the cost of misallocating the asset from 0 to $(1-\theta)(Z - Z_0)$ if $a = s$, and from $(1-p)(Z - Z_0)$ to $(1-p\theta)(Z - Z_0)$ if $s = r$. This is because rehypothecation introduces the model with the risk that the counterparty B loses A's collateral, which occurs with the probability $1 - \theta$.

As a result, the efficiency of rehypothecation is determined by the relative size of

---

[18]In general, A's choice of action can vary with rehypothecation, and we will address this issue in the next section.

these trade-off effects discussed so far. It enhances the provision of funding liquidity to the system so that additional productive investment can be undertaken, B's investment at date 1 in our model, but it introduces the counterparty risk to lose collateral, which may incur the deadweight loss by allocating the asset inefficiently.

5 Conflict between Intermediary's Ex-Post Decision to Rehypothecate and Ex-Ante Efficiency

We have so far assumed that B's (the intermediary's) decision to rehypothecate at date 1 is exogenously given. In this section, we relax this assumption and assume that it is endogenously determined within the model; B can decide whether to rehypothecate A's collateral or not when the investment opportunity arrives at date 1, by comparing the expected return if he rehypothecates A's collateral for the investment versus the expected return if he just keeps it safe until A makes the payment for recovering it from B at a later date. The question is then whether this decision made by B achieves an optimal outcome or not, in other words, whether B's ex post objective is in alignment or conflict with what would be an ex ante efficient decision.

In this section, we build numerical examples that B's and the society's objectives are in conflict. The first example shows that B prefers to rehypothecate even when rehypothecation is inefficient, and the (ex ante) social welfare will be even greater if B can commit not to rehypothecate. The second example shows that B does not rehypothecate even when rehypothecation is efficient , and the social welfare will be even greater if B can commit to rehypothecate.

The direction of this conflict between B's ex post objectives and the ex ante social efficiency depends on terms of the contract between A and B; to be specific, whether the contract involves over-collateralization or under-collateralization from the persepective of A, or equivalently, whether A's private value of collateral, Z, is greater or smaller than the payment for recovering it from B, $X^{\dagger}$.

First, in the case that the contract involves over-collateralization, B tends to be excessively eager to rehypothecate than the socially efficient level. Intuitively, this is because B does not internalize a negative externality that A suffers from not getting back his collateral in case of rehypothecation failure – this negative externality of rehypothecation failure is reflected in the spreads between A's private value on his collateral and

the payment for recovering it, which is positive by the definition of over-collateralization. As a result, without considering this external cost to A when rehypothecation fails, B sometimes chooses to rehypothecate, even when rehypothecation is not socially efficient.

Similarly, in the case that the contract involves under-collateralization, B tends to be excessively cautious to rehypothecate, as B does not internalize a positive externality that A enjoys from not paying the loan when B loses A's collateral in case of rehypothecation failure. Hence, B sometimes chooses not to rehypothecate, even when rehypothecation is socially efficient.

5.1 Over-Collateralization and Excessive Rehypothecation

In this subsection, we build an example in which the contract involves over-collateralization and there arises excessive rehypothecation. We will build this example in parallel with the example from the previous section, picking parameter values that also induce safe behavior in the optimal contract – it is also possible to build examples of under-collateralization and insufficient rehypothecation in cases where the optimal contract induces risky behavior which will be shown latet on. The setup of the example is analogous to that of the models described in the previous sections. For simplicity, we assume that B's outside utility is $J = 0$. In addition, we set the parameter values to satisfy the following additional conditions as well as Assumption $1 \sim 5$.

(i) $Z_0 = 0$

(ii) $\mathcal{B} \equiv R - \frac{b}{1-p} < 0$

(iii) $\theta \in \left(\frac{1}{\theta Y}, \frac{1}{\theta Y} \left(\frac{1 - \mathcal{B}\theta Y}{1 - \mathcal{B}} \right) \right)$ where θY is a fixed positive number.

These conditions are intuitive. First, condition (i) implies that it can never be optimal for A to choose the risky action in any cases, either non-rehypothecation or rehypothecation, so that we can focus on the case in which A takes the safe action. Next, Condition (ii) implies, combined with condition (i), the optimal contract takes the form of over-collateralization. Lastly, condition (iii) implies that B prefers to rehypothecate if $\theta > \frac{1}{\theta Y}$, but rehypothecation becomes efficient only if $\theta > \frac{1}{\theta Y} \left(\frac{1 - \mathcal{B}\theta Y}{1 - \mathcal{B}} \right)$, which is above B's cutoff to participate in rehypothecation by condition (ii).[19] Then, under these assumptions, we can show the following.

---

[19]Note that since the contract involves over-collateralization, i.e., $\mathcal{B} < 0$, B's cutoff where she participates in rehypothecation is smaller than the cutoff where rehypothecation becomes efficient.

Proposition 3. *Under assumptions 1 ~ 5 and (i)-(iii), B prefers to rehypothecate despite the fact that rehypothecation is inefficient,*

Proof. First, note that B prefers to rehypothecate at date 1 if and only if

$$\theta Y(\theta X_s + (1-\theta)Z_0) > X_s \tag{36}$$

where X_s is the payment promised by A for recovering his collateral from B. We can show that under condition (i) and (iii), $Z_0 = 0$ and $\theta > \frac{1}{\theta Y}$, the left hand side of Equation (36) is greater than the right hand side, and thus B wants to rehypothecate.

On the other hand, rehypothecation is not socially efficient if,

$$RI_s - X_s > RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z \tag{37}$$

where I_s, X_s are from Proposition 1 and $I_s^\dagger$, $X_s^\dagger$ are from Proposition 2. The left hand side is A's utility under non-rehypothecation and the right hand side is that under rehypothecation – this equals to the social welfare as we assumed that A has all the bargaining power.

This inequality holds under the parametric restrictions (i), (ii), and (iii). First, note that by substituting the incentive constraints to A's utility function, we can represent it as a function of I_s or $I_s^\dagger$ as follows.

$$RI_s - X_s = (R - \mathcal{B})I_s - Z,$$
$$RI_s^\dagger - \theta X_s^\dagger - (1-\theta)Z = (R - \mathcal{B})I_s^\dagger - Z. \tag{38}$$

Also, by condition (iii), we can show that

$$I_s = \frac{Z}{1-\mathcal{B}} > \frac{\theta^2 Y Z}{1 - \mathcal{B}\theta Y} = I_s^\dagger. \tag{39}$$

Finally, plugging these into I_s and $I_s^\dagger$ in Equation (38), one can derive Inequality (37). □

5.2 Under-Collateralization and Insufficient Rehypothecation

Next, we build an example that the contract between A and B involves under-collateralization and there arises insufficient rehypothecation.

Again, the setup of the example is analogous to that in the previous models. For simplicity, we assume that B's outside utility is $J = 0$. In addition, we make the following restrictions on the parameters beyond Assumption $1 \sim 5$.

(i') $Z_0 = 0$

(ii') $\mathcal{B} \equiv R - \frac{b}{1-p} > 0$.

(iii') $\theta \in \left(\frac{1}{\theta Y} \left(\frac{1 - \mathcal{B}\theta Y}{1 - \mathcal{B}} \right), \frac{1}{\theta Y} \right)$ where θY is a fixed positive number.

As in the previous example, condition (i') implies that choosing the risky action is never optimal for A, and thus we can focus on the case in which A takes the safe action. Condition (ii'), combined with condition (i'), implies that the optimal contract between A and B involves under-collateralization – recall that by the incentive constraint (IC_s''), $X_s < Z$ and $X_s^\dagger < Z$, if and only if $\mathcal{B} < 0$. Lastly, condition (iii') implies rehypothecatio is efficient if $\theta > \frac{1}{\theta Y} \left(\frac{1 - \mathcal{B}\theta Y}{1 - \mathcal{B}} \right)$, but B wants to rehypothecates only if $\theta > \frac{1}{\theta Y}$, which is above the cutoff where rehypothecation is efficient. Then, under these assumptions, we can show the following.

Proposition 4. *Under assumptions $1 \sim 5$ and (i')-(iii'), B does not want to rehypothecate despite the fact that rehypothecation is efficient.*

Proof. First, note that if conditions (i') and (iii') hold, i.e., $Z_0 = 0$ and $\theta > \frac{1}{\theta Y}$, for all X_s, the following inequality holds,

$$\theta Y \left(\theta X_s + (1 - \theta) Z_0 \right) < X_s \tag{40}$$

which implies that the expected revenue from rehypothecation, which is on the left hand side, is smaller than the expected revenue from just keeping collateral, which is on the right hand side, and thus B does not rehypothecate .

Next, we want to show that condition (i'), (ii'), and (iii') imply that rehypothecation is efficient,

$$RI_s - X_s < RI_s^\dagger - \theta X_s^\dagger - (1 - \theta) Z \tag{41}$$

where I_s, X_s are from Proposition 1, and $I_s^\dagger$, $X_s^\dagger$ are from Proposition 2. And, the left hand side is A's expected utility under non-rehypothecation and the right hand side is that under rehypothecation.

To show this inequality, we first plug the incentive constraints (IC'_s) and (IC''_s) into A's utility function, so that represent it as a function of only I_s and $I_s^\dagger$, respectively.

$$RI_s - X_s = (R - \mathcal{B})I_s - Z,$$
$$RI_s^\dagger - \theta X_s^\dagger - (1 - \theta)Z = (R - \mathcal{B})I_s^\dagger - Z. \tag{38}$$

Then, we can compare the welfare in each case by simply comparing I_s with $I_s^\dagger$. Also, by condition (iii'), we can show that

$$I_s = \frac{Z}{1 - \mathcal{B}} < \frac{\theta^2 Y Z}{1 - \mathcal{B}\theta Y} = I_s^\dagger. \tag{42}$$

Finally, plugging these into I_s and $I_s^\dagger$ in Equation (38), Inequality (41) is derived.

$\square$

6 Conclusion

We develop a model of the economic underpinnings of rehypothecation. Rehypothecation is the re-use of the same collateral to support multiple transactions. Rehypothecation helps providing more funding liquidity into the system by allowing the lender to use her borrower's collateral sitting idle in her account for another productive investment. However, it can incur deadweight cost of misallocating the asset when the lender fails having repledged the borrower's collateral to the third party to whom the collateral is less likely to be as valuable to the initial owner.

We show that the individuals' incentives to participate in rehypothecation may not be aligned with economic efficiency. In other words, cases arise in which agents choose socially inefficient levels of rehypothecation. Importantly, the direction of the inefficiency depends on terms of the agents' contracts. If the contract involves over-collateralization, the intermediary tends to be overly eager to rehypothecate; if the contract involves under-collateralization, the intermediary tends to be overly cautious to rehypothecate.

Several natural extensions are worth consideration. Thus far we have not incorporated insurance motives, random fluctuation in the value of the collateral, and the effects of aggregate and idiosyncratic shocks. These important practical considerations are left for future research.

References

Andolfatto, D., Martin, F. M., and Zhang, S. (2015). Rehypothecation and liquidity. *FRB St Louis Paper No. FEDLWP2015-003*.

Biais, B., Heider, F., and Hoerova, M. (2015). Risk-sharing or risk-taking? counterparty risk, incentives and margins. *Journal of Finance, Forthcoming*.

Bolton, P. and Oehmke, M. (2015). Should derivatives be privileged in bankruptcy? *The Journal of Finance*, 70(6):2353–2394.

Bottazzi, J.-M., Luque, J., and Páscoa, M. R. (2012). Securities market theory: Possession, repo and rehypothecation. *Journal of Economic Theory*, 147(2):477–500.

Dang, T. V., Gorton, G., and Holmström, B. (2013). The information sensitivity of a security. *Unpublished working paper, Yale University*.

Donaldson, J. R. and Micheler, E. (2015). Resaleable debt and systemic risk.

Duffie, D., Scheicher, M., and Vuillemey, G. (2015). Central clearing and collateral demand. *Journal of Financial Economics*, 116(2):237–256.

Eren, E. (2014). Intermediary funding liquidity and rehypothecation as determinants of repo haircuts and interest rates. In *27th Australasian Finance and Banking Conference*.

Geanakoplos, J. (2010). The leverage cycle. In *NBER Macroeconomics Annual 2009, Volume 24*, pages 1–65. University of Chicago Press.

Gertler, M., Kiyotaki, N., et al. (2010). Financial intermediation and credit policy in business cycle analysis. *Handbook of monetary economics*, 3(3):547–599.

Gertler, M., Kiyotaki, N., and Queralto, A. (2012). Financial crises, bank risk exposure and government financial policy. *Journal of Monetary Economics*, 59:S17–S34.

Gorton, G. and Pennacchi, G. (1990). Financial intermediaries and liquidity creation. *The Journal of Finance*, 45(1):49–71.

Gorton, G. B., Lewellen, S., and Metrick, A. (2012). The safe-asset share. Technical report, National Bureau of Economic Research.

Gorton, G. B. and Ordonez, G. (2013). The supply and demand for safe assets. Technical report, National Bureau of Economic Research.

Greenwood, R. M., Hanson, S. G., and Stein, J. C. (2010). A comparative-advantage approach to government debt maturity. *Harvard Business School Finance Working Paper*, (1680604).

Holmstrom, B. (2015). Understanding the role of debt in the financial system.

Holmstrom, B. and Tirole, J. (1996). Private and public supply of liquidity. Technical report, National Bureau of Economic Research.

Holmström, B. and Tirole, J. (2011). *Inside and outside liquidity*. MIT press.

Infante, S. (2014). Money for nothing: The consequences of repo rehypothecation. *Available at SSRN*, (2460979).

Kahn, C. M. and Roberds, W. (2007). Transferability, finality, and debt settlement. *Journal of Monetary Economics*, 54(4):955–978.

Koeppl, T. V. (2013). The limits of central counterparty clearing: Collusive moral hazard and market liquidity. Technical report, Queen's Economics Department Working Paper.

Krishnamurthy, A. and Vissing-Jorgensen, A. (2013). Short-term debt and financial crises: What we can learn from us treasury supply. *unpublished, Northwestern University, May*.

Lee, J. (2015). Collateral circulation and repo spreads. *Available at SSRN 2548209*.

Maurin, V. (2014). Re-using the collateral of others: A general equilibrium model of rehypothecation. Technical report, Working paper.

Mills, D. C. and Reed, R. R. (2008). Default risk and collateral in the absence of commitment. *Available at SSRN 1148963*.

Mills, Jr., D. C. and Reed, R. R. (2012). Default risk and collateral in the absence of commitment. *manuscript*.

Monnet, C. (2011). Rehypothecation. *Business Review, Federal Reserve Bank of Philadelphia*, (Q4):18–25.

Moore, J. (2011). Leverage stacks and the financial system. *Ross Prize Lecture*.

Muley, A. (2016). Collateral reuse in shadow banking and monetary policy. *MIT Working Paper*.

Oehmke, M. (2014). Liquidating illiquid collateral. *Journal of Economic Theory*, 149:183–210.

Singh, M. (2010). The sizable role of rehypothecation in the shadow banking system. *IMF Working Papers*.

Singh, M. (2011). Velocity of pledged collateral: analysis and implications. *IMF Working Papers*, pages 1–24.

Singh, M. (2014). *Collateral and Financial Plumbing*. Risk Books.