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Minimum-Variance Kernels, Economic Risk Premia,
and Tests of Multi-beta Models

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Minimum-Variance Kernels, Economic Risk Premia, and Tests of Multi-beta Models

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Abstract: This paper uses minimum-variance (MV) admissible kernels to estimate risk premia associated with economic risk variables and to test multi-beta models. Estimating risk premia using MV kernels is appealing because it avoids the need to 1) identify all relevant sources of risk and 2) assume a linear factor model for asset returns. Testing multi-beta models in terms of restricted MV kernels has the advantage that 1) the candidate kernel has the smallest volatility and 2) test statistics are easy to interpret in terms of Sharpe ratios. The authors find that several economic variables command significant risk premia and that the signs of the premia mostly correspond to the effect that these variables have on the risk-return trade-off, consistent with the implications of the intertemporal capital asset pricing model (I-CAPM). They also find that the MV kernel implied by the I–CAPM, while formally rejected by the data, consistently outperforms a pricing kernel based on the size and book-to-market factors of Fama and French (1993).

JEL classification: G12

Key words: minimum-variance kernels, intertemporal capital asset pricing model, economic risk premia

Introduction

There is considerable evidence against the static Capital Asset Pricing Model (CAPM), suggesting that variables other than the rate of return on a market-portfolio proxy command significant risk premia. The theory of the intertemporal CAPM (I-CAPM, Merton, 1973) suggests that these additional variables should proxy for the position of the investment opportunity set. Hence, beginning with Chen, Roll, and Ross (1986), several researchers have investigated the empirical performance of multi-beta models with macroeconomic risk factors.

Despite significant amount of work on these models, the implications regarding the size and significance of risk premia, for both traded and non-traded risk variables, are less than conclusive. For example, Chen, Roll, and Ross (1986) find that the exposure to the rate of return on the value-weighted NYSE index commands a negative and insignificant risk premium. On the other hand, Burmeister and McElroy (1988) find that exposure to market risk commands a positive and insignificant premium; whereas McElroy and Burmeister (1988) find that the sign of the market premium changes depending on whether a January dummy is included. Finally, Ferson and Harvey (1991) estimate a market risk premium which is generally positive, and in one case significant. When it comes to non-traded, macro-economic variables, such as the inflation rate, Chen, Roll, and Ross estimate negative and often significant risk premia on unexpected inflation. McElroy and Burmeister obtain negative and significant estimates of the unexpected-inflation premium; whereas Burmeister and McElroy obtain positive and significant estimates. Ferson and Harvey obtain estimates that are negative and only marginally significant. In addition, the magnitudes of the estimated risk premia change substantially from one study to the other.

The evidence on the ability of multi-beta models to correctly price asset returns is equally mixed. For example, both McElroy and Burmeister (1988) and Burmeister and McElroy (1988) fail to reject the restrictions of a multi-beta model which uses various macroeconomic variables as factors. On the other hand, Chan, Karceski, and Lakonishok (1998) find that except for the default premium and the term premium, macroeconomic factors perform

poorly in explaining the cross-section of asset returns. Conversely, Ferson and Harvey (1999) claim that loadings on macro variables which predict asset returns, provide significant cross-sectional explanatory power.

This paper contributes to the existing literature by separating the estimation of risk premia from the testing of multi-beta models, and by proposing a new approach for the estimation of risk premia and the testing of multi-beta models.

Our focus is mainly on economic or non-traded risk variables. Hence, risk premia cannot be directly estimated from asset returns, but need to be identified through an asset-pricing model. The model takes the form of a pricing kernel, i.e. a random variable that assigns prices to cash flows to be received in different states of the world (see, for example, Hansen and Richard, 1987). Since markets are generally incomplete, there is a multiplicity of models, i.e. pricing kernels, that are consistent with observed prices. Among these multiple kernels, we follow Hansen and Jagannathan (1991) (HJ) and choose the one with minimum-variance, the MV kernel. We then estimate the risk premia assigned by the MV kernel. Relative to the to the traditional approach of estimating risk premia in the context of a multi-beta model, our method has the advantage that we do not need to identify all relevant sources of risk, nor we need to assume the linearity of returns in the factors. Moreover, since the kernel that we employ is the one with minimum-variance, there is an advantage in terms of the precision of risk-premium estimates.

We then show that a multi-beta model can be translated into an MV pricing kernel, constructed using the cash-flows of the portfolios hedging the economic risk variables driving returns. The resulting kernel has the minimum variance among all the kernels that share the same pricing implications. This property is useful in GMM tests of Euler equations, which tend to "reward" the variability of a kernel (Hansen and Jagannathan, 1997). Moreover, this formulation of a multi-beta model makes the comparison of pricing-kernel volatilities (HJ) and the Hansen-Jagannathan distance measure (Hansen and Jagannathan, 1997) intuitively interpretable in terms of comparisons of Sharpe ratios.

The main empirical results of the paper can be summarized as follows.

First, several of the variables used in previous studies of multi-beta models these variables indeed command significant risk premia. And the sign of the risk premia is generally consistent with the intuition of the intertemporal CAPM (I-CAPM): variables affecting positively (negatively) the risk-return trade off command a positive (negative) risk premium. Namely, the rate of return on a stock market proxy, per-capita consumption growth, the slope of the term structure, the real rate of interest, and the default premium, all command positive (and mostly significant) risk premia. With the exception of the real rate of interest, all these variables have an overall positive effect on future expected returns, and a negative effect on future volatility. The rate of inflation, on the other hand, commands a significant negative premium, consistent with its overall negative effect on expected returns and its positive effect on volatility.

Second, although all the multi-beta models that we consider are formally rejected, their performance differs greatly. The consumption-oriented CAPM (C-CAPM) and the standard CAPM (S-CAPM) have similar performance and are strongly rejected. The I-CAPM and the Fama-French model (FF) are rejected far less strongly, and the I-CAPM consistently outperforms the FF model.

The issues and techniques of this paper are related to several recent papers in the asset-pricing literature. Balduzzi and Kallal (1997) also estimate risk premia assigned by the minimum-variance kernel, but the focus of their analysis is very different. They show that if a candidate pricing kernel assigns risk premia which differ from those of the minimum-variance kernel, then the variability of the candidate must exceed the variability of the minimum-variance kernel. This intuition is exploited to tighten the variance bounds of Hansen and Jagannathan (1991). These new tightened bounds are used to test various versions of the C-CAPM. Lamont (2001) shows that linear factor models which use as regressors hedging portfolio returns, rather than the raw factors, provide more precise estimates of the factor sensitivities. His analysis differs from ours in that he focuses on the sensitivities of asset returns to the factors, rather than on the factor risk premia, and he does not perform tests of the factor models. Jagannathan and Wang (2001) contrast the formulation of multi-beta models in terms of pricing kernel, or stochastic discount factor (SDF), and in terms of a linear

factor model, or beta representation. They show that when the moments of the factors are not known, as it is typically the case, the precision of risk premium estimates for economic factors is the same across the two methods. Moreover, they show that the two methods have a similar ability in detecting mispricing. Their results, together with the generality of the SDF representation, motivate our focus on pricing kernels.

This paper is organized as follows. Section I illustrates the risk premia assigned by the MV admissible kernel. Section II discusses the relation with the analysis of multi-beta models. Section III illustrates the methods used for estimation and testing. Section IV describes the data. Section V presents the empirical results. Finally, Section VI concludes.

I.   MV Kernels and Economic Risk Premia

In this section: i) we define the economic risk premia; ii) we introduce the MV kernel; iii) we calculate the risk premia assigned by the MV kernel; iv) we relate these premia to the excess cash-flows of hedging portfolios; and v) we discuss the effects on risk premia estimates of imposing the positivity constraint on the MV kernel.

A.   Economic Risk Premia

We denote with $\mathbf{r}_{t+1}$ the $N \times 1$ vector of (gross) security returns on the N risky assets under consideration. By the law of one price, we have

$$E_t(\mathbf{r}_{t+1}m_{t+1}) = \mathbf{1} \tag{1}$$

for some admissible stochastic discount factor, or pricing kernel m_{t+1}, where $\mathbf{1}_N$ is an $N \times 1$ vector of ones. In the analysis that follows we interpret all quantities as real quantities. Hence, m_{t+1} stands for the real pricing kernel.

We denote with r_{ft} the possibly time-varying rate of return on a risk-free asset. We have

$$E_t(m_{t+1}) = 1/r_{ft} \tag{2}$$

4

and

$$E_t(r_{ft}m_{t+1}) = 1. \tag{3}$$

For convenience, we perform the analysis that follows in terms of the pricing kernel scaled by the risk-free rate, or the normalized pricing kernel, $r_{ft}m_{t+1} \equiv q_{t+1}$.[1] The normalized kernel q_{t+1} has mean one and satisfies

$$E_t(\mathbf{r}_{t+1}q_{t+1}) = \mathbf{1}_N r_{ft}. \tag{4}$$

In other words, after "adjusting" for q_{t+1}, any expected asset return equals the risk-free rate.[2] One advantage of considering the normalized kernel q_{t+1}, as opposed to the original kernel m_t, is that we do not need to worry about the mean of q_{t+1}, which is set at 1. Moreover, several asset-pricing models, such as the CAPM, have implications for asset premia, but not for the risk-free rate, i.e. they have implications for q_{t+1} but not m_{t+1}.

Using equation (4), we obtain the familiar orthogonality condition

$$E_t[(\mathbf{r}_{t+1} - r_{ft}\mathbf{1}_N)q_{t+1}] = 0. \tag{5}$$

Rearranging, we obtain

$$[E_t(\mathbf{r}_{t+1}) - \mathbf{1}_N r_{ft}]E_t(q_{t+1}) = E_t(\mathbf{r}_{t+1}) - \mathbf{1}_N r_{ft} = -\text{Cov}_t(\mathbf{r}_{t+1}, q_{t+1}). \tag{6}$$

Equation (6) above states that the expected return on any asset in excess of the risk-free rate equals minus the conditional covariance between the asset return and the normalized pricing kernel.

We denote with $\mathbf{y}_{t+1}$ the $K \times 1$ vector of K economic risk variables under consideration. We start by assuming that there exist portfolios exactly mimicking the economic risk variables, and we denote by $\mathbf{r}_{yt+1}$ the $K \times 1$ vector of their returns. We have

$$\mathbf{r}_{yt+1} - \mathbf{1}_N r_{ft} = \lambda_t + [\mathbf{y}_{t+1} - E_t(\mathbf{y}_{t+1})], \tag{7}$$

---

[1]Other papers that focus on the normalized kernel q_{t+1}, rather than m_{t+1}, are, for example, Balduzzi and Kallal (1997) and Dumas and Solnik (1995).

[2]This means that $q_{t+1} = \theta_{t+1}/\theta_t$, where θ_t is a valid martingale pricing process.

where λ_t is a $K \times 1$ coefficient vector. According to (6), we have

$$\lambda_t = E_t(\mathbf{r}_{y,t+1} - \mathbf{1}_N r_{ft}) = -\mathrm{Cov}_t(\mathbf{y}_{t+1}, q_{t+1}), \tag{8}$$

which is the vector of equilibrium conditional risk premia on the corresponding economic risk variables, $\mathbf{y}_t$. The unconditional risk premia are obtained by applying the law of iterated expectations to (8):

$$\lambda \equiv \mathrm{E}(\lambda_t) = \mathrm{E}(\mathbf{r}_{y,t+1}) - \mathbf{1}_N \mathrm{E}(r_{ft}) = -\mathrm{Cov}_t(\mathbf{y}_{t+1}, q_{t+1}). \tag{9}$$

Note that, if the economic risk variables have conditional standard deviation equal to one, then the conditional risk premia also have the interpretations of Sharpe ratios: mean excess returns per unit of risk. Similarly, the unconditional risk premia also have the interpretation of mean Sharpe ratios.

From the discussion above it follows that if the risk variables are traded, then any admissible pricing kernel will assign the same risk premia. Indeed, in this case the risk variables would be payoffs on existing assets, and all admissible pricing kernels correctly price existing assets. The estimation of the risk premia on traded variables would boil down to the estimation of mean cash flows on traded securities.

On the other hand, if the risk variables are not traded, then the risk premia λ_t are not immediately available. The present paper focuses on this situation. This is the reason why the risk variables $\mathbf{y}_{t+1}$ are denoted "economic," to be differentiated from traded variables. There are both academic and practical reasons why we want to estimate risk premia associated with non-traded risks. First, these risk premia are an indication of how "important" an economic variable is. Second, if new securities are introduced whose payoffs track the economic variables, it is important to know how the new securities should be priced.

B.   The MV Kernel

Following HJ, we can construct an admissible (normalized) pricing kernel, i.e. a random variable with mean of one that satisfies equation (4) and that is linear in $\mathbf{r}_{t+1}$: $q^*_{t+1} \equiv$

$\alpha_{0t} + \mathbf{r}_{t+1}^\top \alpha_t$, where α_{0t} is a scalar and α_t is an $N \times 1$ coefficient vector. We have

$$E_t(\mathbf{r}_{t+1})E_t(q_{t+1}) + \mathrm{Cov}_t(\mathbf{r}_{t+1}, q_{t+1}) = \mathbf{1}_N r_{ft}. \tag{10}$$

Using (4) and $q_{t+1}^* \equiv \alpha_{0t} + \mathbf{r}_{t+1}^\top \alpha_t$, we have

$$-\mathbf{\Sigma}_{rrt}\alpha_t = E_t(\mathbf{r}_{t+1}) - r_{ft}\mathbf{1}_N, \tag{11}$$

where $\mathbf{\Sigma}_{rrt}$ is the covariance matrix of risky asset returns (which we assume to be invertible). We obtain

$$\alpha_t = -\mathbf{\Sigma}_{rrt}^{-1}[E_t(\mathbf{r}_{t+1}) - r_{ft}\mathbf{1}_N], \tag{12}$$

while

$$\alpha_{0t} = 1 - E_t(\mathbf{r}_{t+1})^\top \alpha_t. \tag{13}$$

HJ show that q_{t+1}^* has the minimum variance among all the admissible kernels.

The MV kernel q_{t+1}^* has at least two other properties worth recalling. First, the vector α_t is proportional to the vector of portfolio weights of the tangency portfolio obtained from the risky-security returns $\mathbf{r}_{t+1}$. Hence, q_{t+1}^* is perfectly, negatively, correlated with the rate of return on the tangency portfolio, $r_{\tau t+1}$. Another important property is that the conditional variance of q_{t+1}^* equals the squared conditional Sharpe ratio of the tangency portfolio, $S_{\tau t}$. In fact, given perfect correlation between $r_{\tau t+1}$ and q_{t+1}^*, we have

$$[\mathrm{Cov}_t(r_{\tau,t+1}, q_{t+1}^*)]^2 = \mathrm{Var}_t(r_{\tau,t+1})\mathrm{Var}_t(q_{t+1}^*). \tag{14}$$

Since q_t^* prices correctly all the securities under consideration, it also prices correctly the tangency portfolio, and we have $-\mathrm{Cov}_t(r_{\tau,t+1}, q_{t+1}^*) = E_t(r_{\tau,t+1}) - r_{ft}$. Using this result and rearranging equation (14) above, we obtain

$$\left[\frac{E_t(r_{\tau,t+1}) - r_{ft}}{\sqrt{\mathrm{Var}_t(r_{\tau,t+1})}}\right]^2 \equiv S_{\tau t}^2 = \mathrm{Var}_t(q_{t+1}^*). \tag{15}$$

Finally, since $\mathrm{Var}_t(q_{t+1}^*) = E_t[(q_{t+1} - 1)^2]$, then $\mathrm{Var}(q_{t+1}^*) = \mathrm{E}[(q_{t+1} - 1)^2]$, and

$$\mathrm{Var}(q_{t+1}^*) = \mathrm{E}(S_{\tau t}^2). \tag{16}$$

Hence, the unconditional variance of q_{t+1}^* equals the mean squared Sharpe ratio of the tangency portfolio.

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C. The MV Kernel and Economic Risk Premia

As mentioned earlier, since the economic risk variables $\mathbf{y}_{t+1}$ are typically not traded, the associated risk premia differ depending on the model used to value securities. This generates a problem since, unless markets are complete, there is a multiplicity of pricing kernels which are admissible, and there is no obvious reason to choose one kernel over another. Our approach to this problem is to choose the most parsimonious admissible pricing kernel, i.e. the one with the lowest variability. Given the MV kernel q_{t+1}^*, the risk premia assigned by q_{t+1}^* are given by

$$\lambda_t^* \equiv -\mathrm{Cov}_t(\mathbf{y}_{t+1}, q_{t+1}^*) \tag{17}$$

and

$$
\begin{aligned}
\lambda^* &\equiv -\mathrm{E}[\mathrm{Cov}_t(q_{t+1}^* \mathbf{y}_{t+1})] \\
&= -\mathrm{Cov}(\mathbf{y}_{t+1}, q_{t+1}^*).
\end{aligned}
\tag{18}
$$

The discussion of the previous section highlights one first reason to focus on the risk premia assigned by the MV admissible kernel. The MV kernel is perfectly correlated with the rate of return on the tangency portfolio. Hence, if the MV kernel assigns a significant risk premium to an economic risk variable, this means that the rate of return on the tangency portfolio correlates significantly with that variable. Since the tangency-portfolio return is the natural benchmark against which all other portfolios are evaluated, this also means that the economic variable is of relevance to the investors.

D. Risk Premia and Hedging Portfolios

While the risk premia λ_t^* are completely characterized in terms of the interaction between the economic risk variables and the MV kernel, an alternative characterization in terms of hedging portfolios turns out to be useful. Namely, consider the conditional projections of the economic risk variables $\mathbf{y}_{t+1}$ onto the augmented span of returns, $\mathbf{y}_{t+1}^* \equiv \gamma_{0t} + \gamma_t^\top \mathbf{r}_{t+1}$, where

γ_{0t} is a $K \times 1$ vector and γ_t is an $N \times K$ matrix. Let

$$\lambda_t^* \equiv -\text{Cov}_t(\mathbf{y}_{t+1}^*, q_{t+1}) = -\text{Cov}_t(\mathbf{y}_{t+1}^*, q_{t+1}^*). \tag{19}$$

Since q_{t+1}^* satisfies the moment restriction (4) by construction, we have

$$
\begin{aligned}
\lambda_t^* &= -\text{Cov}_t(\mathbf{y}_{t+1}^*, q_{t+1}^*) \\
&= -\text{Cov}_t(\gamma_t^\top \mathbf{r}_{t+1}, q_{t+1}^*) \\
&= -\gamma_t^\top \text{Cov}_t(\mathbf{r}_{t+1}, q_{t+1}^*) \\
&= \gamma_t^\top [E_t(\mathbf{r}_{t+1}) - \mathbf{1}_N r_{ft}], \tag{20}
\end{aligned}
$$

which is the vector of mean cash flows generated by the portfolios hedging the economic risk variables financed at the riskless rate. In other words, λ_t^* is the vector of risk premiums on the hedging portfolios for the economic risk variables. This implies that λ_t^* does not depend on the choice of the (normalized) pricing kernel, but only on the asset returns under scrutiny.

Note that the hedging portfolios defined here are analogous to the "economic tracking portfolios" of Lamont (2001). The main difference in his approach is that the portfolios are constructed to track changes in expectations of future realizations of the economic variables. Instead, our hedging portfolios are designed to track the contemporaneous realizations of economic variables.

The composition of the hedging portfolios is worth further discussion. First, the hedging portfolios contain allocations to the riskless asset in the amount γ_{0t}/r_{ft}. (The rates of return on the hedging portfolios contain a constant component γ_{0t} resulting from the investment in the riskless asset.) Second, the hedging portfolio quantities do not sum up to one. Third, the composition of the hedging portfolios corresponds to the coefficients of a regression of the economic risk variables on the asset returns:

$$
\begin{aligned}
\gamma_t &= \mathbf{\Sigma}_{rrt}^{-1} \mathbf{\Sigma}_{ryt} \\
\gamma_{0t} &= E_t(\mathbf{y}_{t+1}) - \gamma_t^\top E_t(\mathbf{r}_{t+1}). \tag{21}
\end{aligned}
$$

Hence, the hedging portfolios are the discrete-time counterparts of the portfolios held by a dynamic portfolio optimizer to hedge against changes in the investment-opportunity set (see Merton (1973)).

The introduction of the hedging portfolios and their cash flows also provides an additional motive of interest in the risk premia λ_t^*. If one were to introduce a new security whose payoffs track an economic risk variable y_{kt+1}, one can place bounds on its expected rate of return in excess of the risk-free rate, λ_{kt}. As shown in Balduzzi and Kallal (1997), these bounds are centered around the expected excess cash flows of the mimicking portfolio, i.e. they are centered around λ_{kt}^*.[3]

Two issues concerning the estimation and interpretation of the risk premia are also worth noting. First, the excess cash flows on the hedging portfolios in general differ from minus the cross products between the MV kernel, in excess of its mean, and the risk factors. In fact, we have

$$\gamma_t^\top(\mathbf{r}_{t+1} - \mathbf{1}_N r_{ft}) = \mathbf{\Sigma}_{yrt}\mathbf{\Sigma}_{rrt}^{-1}(\mathbf{r}_{t+1} - \mathbf{1}_N r_{ft}), \tag{22}$$

whereas

$$-\mathbf{y}_{t+1}(q_{t+1}^* - 1) = \mathbf{y}_{t+1}[\mathbf{r}_{t+1} - E_t(\mathbf{r}_{t+1})]^\top \mathbf{\Sigma}_{rrt}^{-1}[E_t(\mathbf{r}_{t+1}) - \mathbf{1}_N r_{ft}]. \tag{23}$$

The variances of the two quantities also differ, which means that their expectations will be estimated with different degrees of precision. The variance of the excess cash-flows on the mimicking portfolios is given by

$$
\begin{aligned}
\mathrm{Var}_t[\gamma_t^\top(\mathbf{r}_{t+1} - \mathbf{1}_N r_{ft})] &= \mathrm{Var}_t(\gamma_t^\top \mathbf{r}_{t+1}) \\
&= \mathbf{\Sigma}_{yrt}\mathbf{\Sigma}_{rrt}^{-1}\mathbf{\Sigma}_{rrt}\mathbf{\Sigma}_{rrt}^{-1}\mathbf{\Sigma}_{ryt} \\
&= \mathbf{\Sigma}_{yrt}\mathbf{\Sigma}_{rrt}^{-1}\mathbf{\Sigma}_{ryt}.
\end{aligned} \tag{24}
$$

On the other hand, the variance of the cross product between the MV kernel, in excess of its mean, and the risk factors is given by the expression

$$\mathrm{Var}_t[\mathbf{y}_{t+1}(q_{t+1}^* - 1)] = E_t[\mathbf{y}_{t+1}\mathbf{y}_{t+1}^\top(q_{t+1}^* - 1)^2] - E_t[\mathbf{y}_{t+1}(q_{t+1}^* - 1)]E_t[\mathbf{y}_{t+1}(q_{t+1}^* - 1)]^\top$$

[3]Specifically, they show that

$$\lambda_{kt}^* - \sqrt{[\mathrm{Var}_t(q_{t+1}) - \mathrm{Var}_t(q_{t+1}^*)][\mathrm{Var}_t(y_{kt+1}) - \mathrm{Var}_t(y_{kt+1}^*)]} \le \lambda_{kt}$$

$$\le \lambda_{kt}^* + \sqrt{[\mathrm{Var}_t(q_{t+1}) - \mathrm{Var}_t(q_{t+1}^*)][\mathrm{Var}_t(y_{kt+1}) - \mathrm{Var}_t(y_{kt+1}^*)]},$$

where q_{t+1} is an admissible kernel for the underlying economy. A similar result is obtained in Cochrane and Saá-Requejo (2000) to calculate bounds on the price of an option when markets are incomplete.

$$= E_t[\mathbf{y}_{t+1}\mathbf{y}_{t+1}^\top(q_{t+1}^* - 1)^2] - \lambda_t^*(\lambda_t^*)^\top, \tag{25}$$

which cannot be easily formulated in terms of the moments of economic variables and returns.

Second, the equality between the mean excess cash-flows on the hedging portfolios and the risk premia assigned by the MV kernel allows for a conversion of the risk premia λ_t^* into Sharpe ratios. Specifically, we obtain Sharpe ratios by dividing λ_t^* by the standard deviations of the excess cash flows on the hedging portfolios.[4] Note, though, that the conditional Sharpe ratio of the k-th hedging portfolio, $S_{y_k^* t}$, depends on the risk premium, on the volatility of the economic variable, as well as on how closely the economic variable is replicated. In fact, we have

$$S_{y_k^* t} \equiv \frac{\lambda_{kt}^*}{\sigma_{y_k^*}} = \frac{\lambda_{kt}^*}{\sigma_{y_k}\sqrt{R_{y_k t}^2}}, \tag{26}$$

where $R_{y_k t}^2$ is the R-squared of the conditional projection of y_{kt+1} onto the augmented span of returns. This observation is important because two portfolios hedging variables with the same volatility might receive the same risk premium, and still command different Sharpe ratios. This is because the two portfolios capture different fractions of the variability of the corresponding economic variables.

E.   Positivity

While the minimum-variance pricing kernel q_t^* satisfies the law of one price, equation (5), in general it does not satisfy the no-arbitrage condition $q_t^* > 0$. Nonetheless, as in HJ, we can extend our analysis to take this restriction into account. Indeed, one advantage of our approach is that we can estimate economic risk premia without imposing an explicit asset-pricing model, while at the same time we can impose the no-arbitrage condition which applies to any admissible kernel.

In the following, we want determine whether the imposition of the no-arbitrage condition can have a large impact on our risk-premia estimates. Let $\tilde{\alpha}_t$ denote an $N \times 1$ coefficient

---

[4]This is also a way to obtain quantities that are scale-free, i.e. they are not affected by the fact that the portfolio weights do not sum up to one.

vector, and define $\tilde{q}_{t+1} \equiv (\tilde{\alpha}_{0t} + \mathbf{r}_{t+1}^\top \tilde{\alpha}_t)^+ \equiv \max(\tilde{\alpha}_{0t} + \mathbf{r}_{t+1}^\top \tilde{\alpha}_t, 0)$. Assume

$$E_t(\tilde{q}_{t+1}\mathbf{r}_{t+1}) = \mathbf{1}_N r_{ft}. \tag{27}$$

As shown by HJ, the random variable $\tilde{q}_{t+1}$ has the smallest variance among all nonnegative random variables with mean of one, satisfying restriction (27).

Consider the risk premium $\tilde{\lambda}_{kt}$ assigned by $\tilde{q}_{t+1}$, we can write

$$
\begin{aligned}
\tilde{\lambda}_{kt} &\equiv -E_t[y_{k,t+1}(\tilde{q}_{t+1} - 1)] \\
&= -E_t[y_{k,t+1}^*(\tilde{q}_{t+1} - 1)] - E_t[(y_{k,t+1} - y_{k,t+1}^*)(\tilde{q}_{t+1} - 1)] \\
&= \lambda_{kt}^* - E_t[(y_{k,t+1} - y_{k,t+1}^*)(\tilde{q}_{t+1} - 1)], \tag{28}
\end{aligned}
$$

where, in general, $E_t[(y_{k,t+1} - y_{k,t+1}^*)(\tilde{q}_{t+1} - 1)] \neq 0$. Hence, when the positivity restriction is imposed, the risk premium assigned by the minimum-variance kernel differs from the mean excess cash flow generated by the hedging portfolio by the quantity $-E_t[(y_{k,t+1} - y_{k,t+1}^*)(\tilde{q}_{t+1} - 1)]$. If $\tilde{q}_t$ is volatile, and if y_{kt} is mimicked poorly by its nearest hedge, then there is the potential for the discrepancy to be substantial.

In summary, when the no-arbitrage condition is imposed, the equivalence between the risk premia assigned by the minimum-variance kernel and the mean excess cash flows generated by the mimicking portfolios is no longer valid. Whether the "wedge" introduced between the two quantities is economically relevant is a question that will be addressed in the empirical analysis. It is important to note, though, that our approach of estimating risk premia assigned by MV kernels allows us to avoid the formulation of an explicit asset-pricing model, while at the same time we can impose the no-arbitrage restriction that any equilibrium model should satisfy.

II.   Multi-beta Models

In this section: i) we illustrate the relation between the risk premia assigned by the MV kernel and those assigned by multi-beta models; ii) we show how the restrictions of a multi-

beta model can be translated into the formulation of an MV kernel; and iii) we review four multi-beta models that will be tested in the empirical analysis.

A. Risk Premia and Multi-beta Models

A multi-beta model implies that expected excess returns are linear in the sensitivities of the returns to the economic risk variables, with coefficients given by the risk premia associated with the factors:

$$E_t(\mathbf{r}_{t+1}) - r_{ft}\mathbf{1}_N = \beta_t^\top \lambda_t, \tag{29}$$

where $\beta_t = \mathbf{\Sigma}_{yy}^{-1}\mathbf{\Sigma}_{yr}$ is a $K \times N$ matrix of projection coefficients of the returns onto the economic risk variables. For the multi-beta models to be meaningful, we assume $K < N$: the number of factors driving returns is strictly smaller than the number of assets. Asset returns are described by the model

$$\mathbf{r}_{t+1} - \mathbf{1}_N r_{ft} = \beta_t^\top \lambda_t + \beta_t^\top [\mathbf{y}_{t+1} - E_t(\mathbf{y}_{t+1})] + \mathbf{e}_{t+1}, \tag{30}$$

where $\mathbf{e}_{t+1}$ is a vector of $N \times 1$ mean-zero perturbations orthogonal to the economic risk variables $\mathbf{y}_{t+1}$, with covariance matrix $\mathbf{\Sigma}_{eet}$. The pricing kernel q_{yt+1} underlying the pricing result (29) has the form[5]

$$q_{yt+1} = 1 - [\mathbf{y}_{t+1} - E_t(\mathbf{y}_{t+1})]^\top \mathbf{\Sigma}_{yyt}^{-1}\lambda_t. \tag{31}$$

Consider now the risk premia assigned by the minimum-variance kernel q_t^*. From (20) we have

$$
\begin{aligned}
\lambda_t^* &= \gamma_t^\top \beta_t^\top \lambda_t \\
&= \mathbf{\Sigma}_{yrt}\mathbf{\Sigma}_{rrt}^{-1}\mathbf{\Sigma}_{ryt}\mathbf{\Sigma}_{yyt}^{-1}\lambda_t.
\end{aligned} \tag{32}
$$

[5]Note that the linearity of the pricing kernel in the factors is not a restrictive assumption. In fact, we can define the "factor" $y_{qt+1} \equiv -q_{t+1}$. In this formulation, a single-beta version of equation (29) obtains, where the factor is (minus) the pricing kernel itself.

To make the relation between λ_t^* and λ_t more immediate, note that

$$
\begin{aligned}
\mathrm{Var}_t(\gamma_t^\top \mathbf{r}_{t+1}) &= \mathbf{\Sigma}_{y^*y^*t} \\
&= \mathbf{\Sigma}_{yrt}\mathbf{\Sigma}_{rrt}^{-1}\mathbf{\Sigma}_{rrt}\mathbf{\Sigma}_{rrt}^{-1}\mathbf{\Sigma}_{ryt} \\
&= \mathbf{\Sigma}_{yrt}\mathbf{\Sigma}_{rrt}^{-1}\mathbf{\Sigma}_{ryt}.
\end{aligned}
\tag{33}
$$

Hence, from equation (32) we have

$$
\mathbf{\Sigma}_{y^*y^*t}^{-1}\lambda_t^* = \mathbf{\Sigma}_{yyt}^{-1}\lambda_t.
\tag{34}
$$

Several comments are worth making based on equation (34) above. First, the l.h.s. of (34) is the vector of coefficients of the MV kernel q_{y^*t+1} that prices the mimicking portfolios, $q_{y^*t+1} \equiv \alpha_{0yt} + \alpha_{yt}^\top \mathbf{y}_{t+1}^* = 1 - [\mathbf{y}_{t+1}^* - E_t(\mathbf{y}_{t+1}^*)]^\top \mathbf{\Sigma}_{y^*y^*t}^{-1}\lambda_t^*$, while the r.h.s. of (34) is the vector of coefficients linking the linear kernel q_{yt+1} to the economic factors $\mathbf{y}_{t+1}$ (see equation (31)). Second, if the factors are standardized and made orthogonal, i.e. $\mathbf{\Sigma}_{yyt} = \mathbf{I}_K$, then the risk premia of a multi-beta model coincide with the coefficients of q_{y^*t+1}. Third, from equation (34) we have

$$
\lambda_t = \mathbf{\Sigma}_{yyt}\mathbf{\Sigma}_{y^*y^*t}^{-1}\lambda_t^*.
\tag{35}
$$

Hence, one important difference between λ_t^* and λ_t is that the individual elements of λ_t are interrelated, in that they depend on the choice of factors, while the elements of λ_t^* can be estimated separately without the need to identify all relevant sources of risk. Moreover, equation (35) linking λ_t to λ_t^* is meaningless if expected returns do not conform to the multi-beta model, whereas the vector λ_t^* is of economic importance regardless of the validity of an asset-pricing model.

B.   Tests of Multi-beta Models and MV Kernels

When it comes to testing a multi-beta model, two approaches are typically used. The first approach is that of testing the restriction that a multi-beta model places on the asset-return generation process. Namely, one would test equation (30). The second approach is that of testing the pricing kernel q_{yt}.

Here we propose a third approach. This approach is based on the observation that since the projection of $\mathbf{y}_{t+1}$ onto the augmented span of returns is $\mathbf{y}_{t+1}^*$, equation (34) implies that q_{y^*t+1} is the projection of q_{yt+1} onto the augmented span of returns. Hence, q_{y^*t+1} inherits the pricing properties of q_{yt+1} w.r.t. the span of returns $\mathbf{r}_{t+1}$. This means that testing q_{y^*t} is equivalent to testing q_{yt}.

This alternative approach has two main advantages. First, it reduces the variability of the kernel being tested. This is important in the standard tests of overidentifying restrictions pioneered by Hansen and Singleton (1982). These tests are based on estimates of the statistic

$$\chi^2 \equiv T \times \mathrm{E}[(\mathbf{r}_{t+1} - r_{ft}\mathbf{1}_N)x_{t+1}]^\top \mathrm{Var}[(\mathbf{r}_{t+1} - r_{ft}\mathbf{1}_N)x_{t+1}]^{-1}\mathrm{E}[(\mathbf{r}_{t+1} - r_{ft}\mathbf{1}_N)x_{t+1}], \quad (36)$$

where T is the number of observations and x_{t+1} is the candidate kernel being tested. If x_{t+1} is very volatile, then the estimated statistic χ^2 can be "small" even though the mean pricing errors $\mathrm{E}[(\mathbf{r}_{t+1} - r_{ft}\mathbf{1}_N)x_{t+1}]$ are "large," leading to type-II errors.

The second advantage has to do with the interpretability of tests of asset-pricing models. Hansen and Jagannathan in their 1991 and 1997 articles have developed two tests that complement the tests of overidentifying restrictions. The first test compares the volatility of the candidate kernel x_t to the volatility of the MV kernel q_t^*. Specifically, one tests the condition:[6]

$$\mathrm{HJV} \equiv \sqrt{\mathrm{Var}(x_t)} - \sqrt{\mathrm{Var}(q_t^*)} \geq 0. \quad (37)$$

If the candidate kernel is the MV kernel constructed from the hedging-portfolio cash flows, q_{y^*t}, then its variance is the mean squared Sharpe ratio of the tangency portfolio constructed using the hedging portfolios. In other words, testing $\mathrm{HJV} \geq 0$ is equivalent to testing whether the mean squared Sharpe ratio of the tangency portfolio constructed using the hedging portfolio is greater or equal to the Sharpe ratio of the tangency portfolio constructed using all the securities available:

$$\mathrm{HJV} = \sqrt{\mathrm{E}(S_{y^*t}^2)} - \sqrt{\mathrm{E}(S_{\tau t}^2)} \geq 0. \quad (38)$$

---

[6]See, for example, Cecchetti, Lam, and Mark (1994), and Cochrane and Hansen (1992). Note that we formulate the volatility bound here in terms of standard deviations rather than variances. This is because standard deviations are more easily interpretable than variances.

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Obviously, $\mathrm{E}(S^2_{y^*t}) < \mathrm{E}(S^2_{\tau t})$: restricting the composition of the tangency portfolio to $K < N$ linear combinations of the original N assets reduces the Sharpe ratio. This means that the HJV statistic is always negative and it measures the loss in Sharpe ratio that an investor would suffer for believing in a multi-beta model. The second test is the Hansen-Jagannathan (1997) distance test, which is based on the volatility of the difference between the projection of x_t onto the augmented span of returns, x^*_t, and q^*_t:

$$\mathrm{HJD} \equiv \sqrt{\mathrm{E}(x^*_t - q^*_t)^2}. \tag{39}$$

Note that, if the candidate kernel is q_{y^*t}, its projection onto the augmented span of returns is q_{y^*t} itself. Moreover, $\mathrm{E}(q_{y^*t}) = \mathrm{E}(q^*_t) = 1$, and $\mathrm{Cov}(q_{y^*t}, q^*_t) = \mathrm{Var}(q^*_t)$. Hence, we can rewrite (39) as[7]

$$\mathrm{HJD} = \sqrt{\mathrm{E}(S^2_{\tau t}) - \mathrm{E}(S^2_{y^*t})}. \tag{40}$$

C. Multi-beta Models

We now review asset-pricing theories which impose restrictions on the form of the linear kernel q_{yt+1} of a multi-beta model. We consider four models: the I-CAPM of Merton (1973); the S-CAPM, which is embedded in the I-CAPM; the C-CAPM of Breeden (1979); and the FF model of Fama and French (1993).

Merton's (1973) I-CAPM, in its discrete-time approximate version, implies a pricing kernel q_{yt+1} linear in the market return and the realizations of economic variables driving the investment-opportunity set. Hence, according to the I-CAPM, the projection q_{y^*t+1} is obtained from the cash flows of the portfolios hedging the market and the variables affecting the investment-opportunity set. The S-CAPM is a special case of the I-CAPM, where q_{y^*t+1} can be constructed using the market portfolio only.

Consider now Breeden's (1979) C-CAPM. Its pricing implication, in its discrete-time approximate version, is that q_{yt+1} is linear in aggregate per-capita consumption growth.

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[7]This result is analogous to the one derived by DeRoon and Nijman (2001) for the case where the candidate kernel is the MV kernel constructed using a subset of the asset returns.

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Hence, the projection q^*_{yt+1} is obtained from the cash flows of the portfolio hedging aggregate per-capita consumption growth.[8]

The FF model combines a risk-based explanation for expected excess returns, i.e. the exposure to the market, with characteristic-based explanations, i.e. the total market value and the book-to-market ratio. The characteristics are transformed into portfolios to obtain a multi-beta representation, where expected excess returns are explained by the betas with respect to the market, the size factor, and the book-to-market factor. In the empirical analysis, we test an extended version of the FF model, where two "bond" factors, the term and the default spreads, are added to the three "equity" factors to price the cross-section of both bonds and equities. Hence, the projected kernel q_{y^*t+1} is linear in the cash flows of the three equity factors (they are traded), and in the cash flows of the portfolios hedging the term and default spreads.

III.   Estimation

In this section: i) we explain the approach taken to document time-variation in the first and second moments of asset returns; ii) we illustrate the estimation of the MV kernel; iii) we illustrate the estimation of the economic risk premia; iv) we illustrate how the explicit asset-pricing models discussed in the previous section are tested.

A.   Predictability and Heteroskedasticity

Without loss of generality, we assume the first element of the instrument vector $\mathbf{z}_t$ to be unity, $z_{1t} = 1$. We model the conditional mean and conditional volatility of asset returns as linear functions of the instruments. Namely, we assume

$$E_t(\mathbf{r}_{t+1}) \;=\; \mu_{k1} + \mu_{k2}z_{2t} + \ldots + \mu_{kJ}z_{Jt}$$

---

[8]Breeden, Gibbons, and Litzenberger (1989) exploit a similar result, testing the mean-variance efficiency of the portfolio that tracks per-capita consumption growth.

$$E_t[|\mathbf{r}_{t+1} - E_t(\mathbf{r}_{t+1})|] = \sigma_{k1} + \sigma_{k2}z_{2t} + \ldots + \sigma_{kJ}z_{Jt}. \tag{41}$$

B.   The MV Kernel

As is common in the asset-pricing literature (see, for example, Cochrane and Hansen, 1992), we consider "scaled" versions of the two conditions that an admissible kernel must satisfy:

$$\mathbf{z}_t E_t(q^*_{t+1}) = \mathbf{z}_t \tag{42}$$

$$E_t[(\mathbf{r}_{t+1} \otimes \mathbf{z}_t)q^*_{t+1}] = (\mathbf{1}_N \otimes \mathbf{z}_t)r_{ft}. \tag{43}$$

Assuming stationarity, and applying the law of iterated expectations, we have

$$\mathrm{E}(\mathbf{z}_t q^*_{t+1}) = \mathrm{E}(\mathbf{z}_t) \tag{44}$$

$$\mathrm{E}[(\mathbf{r}_{t+1} \otimes \mathbf{z}_t)q^*_{t+1}] = \mathrm{E}[(\mathbf{1}_N \otimes \mathbf{z}_t)r_{ft}]. \tag{45}$$

The two conditions above ensure that, conditioning on $\mathbf{z}_t$, q^*_t has mean one and correctly prices the securities under consideration. Hence, we proceed to construct such a random variable following HJ. We define $\mathbf{r}^z_{t+1} \equiv \mathbf{r}_{t+1} \otimes \mathbf{z}_t$ and $\mathbf{1}^z_t \equiv \mathbf{1}_N \otimes \mathbf{z}_t$. We have $q_{t+1} \equiv \mathbf{z}_t^\top \alpha_0 + (\mathbf{r}^z_{t+1})^\top \alpha$, where α_0 is a $J \times 1$ coefficient vector and α is a $NJ \times 1$ coefficient vector. The two coefficient vectors are estimated by method of moments, imposing the conditions (44)-(45).[9] When the positivity constraint is imposed, we estimate the kernel $\tilde{q}_{t+1} \equiv [\mathbf{z}_t^\top \tilde{\alpha}_0 + (\mathbf{r}^z_{t+1})^\top \tilde{\alpha}]^+$ satisfying (44)-(45).

C.   Economic Risk Premia

In order to estimate the conditional risk premia associated with the variables y_{kt} we use two approaches. First, we consider the conditional covariance between the minimum-variance

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[9]Closed-form expressions for the two coefficient vectors are given by:

$$\alpha = -\{\mathrm{E}[\mathbf{r}^z_{t+1}(\mathbf{r}^z_{t+1})^\top] - \mathrm{E}(\mathbf{r}^z_{t+1}\mathbf{z}_t^\top)\mathrm{E}(\mathbf{z}_t\mathbf{z}_t^\top)^{-1}\mathrm{E}[\mathbf{z}_t(\mathbf{r}^z_{t+1})^\top]\}^{-1}\{\mathrm{E}(\mathbf{r}^z_{t+1}\mathbf{z}_t^\top)\mathrm{E}(\mathbf{z}_t\mathbf{z}_t^\top)^{-1}\mathrm{E}(\mathbf{z}_t) - \mathrm{E}(r_{ft}\mathbf{1}^z_t)\}$$

$$\alpha_0 = \mathrm{E}(\mathbf{z}_t\mathbf{z}_t^\top)^{-1}\{\mathrm{E}(\mathbf{z}_t) - \mathrm{E}[\mathbf{z}_t(\mathbf{r}^z_{t+1})^\top]\alpha\}.$$

kernel and y_k. Second, we construct hedging portfolios and we estimate their conditional
risk premia.

In implementing the first approach, we assume

$$\lambda_{kt}^* \equiv -\text{Cov}_t(y_{k,t+1}, q_{t+1}^*) = -E_t[y_{k,t+1}(q_{t+1}^* - 1)] = \lambda_{k1} + \lambda_{k2}z_{2t} + \ldots + \lambda_{kJ}z_{Jt}. \quad (46)$$

Without loss of generality, we assume $\text{Var}(z_{jt}) = 1$. Hence the coefficients λ_{kj} can be
interpreted as the change in the conditional risk premium for a one-standard-deviation change
in the instrument. The assumption that the conditional risk premia are determined by the
set of instruments $\mathbf{z}_t$ is quite natural: the conditional risk premia assigned by the minimum-
variance kernel are the expected cash flows generated by the hedging portfolio financed at
the riskless rate. Hence, if the variables in $\mathbf{z}_t$ are predictors of asset returns, they should also
predict excess returns on the hedging portfolios. In fact, modeling the time-variation of risk
premia in this fashion is common to other studies, Ferson and Harvey (1991), for example.
Note that, without loss of generality, we can also assume $\text{E}(z_{jt}) = 0$, $i = 2, \ldots, J$. This
means that

$$\text{E}(\lambda_{kt}^*) = \lambda_{k1}. \quad (47)$$

The distinction between conditional and unconditional risk premia is important, because,
even if the unconditional premium is close to zero, the conditional premia may take values
over time which are significantly different from zero.

When the positivity restriction is imposed, we have

$$\tilde{\lambda}_{kt} \equiv -\text{Cov}_t(y_{k,t+1}, \tilde{q}_{t+1}) = -E_t[y_{k,t+1}(\tilde{q}_{t+1} - 1)] = \tilde{\lambda}_{k1} + \tilde{\lambda}_{k2}z_{2t} + \ldots + \tilde{\lambda}_{kJ}z_{Jt}. \quad (48)$$

The second approach to the estimation of the economic risk premia is based on the
construction of hedging portfolios. The cash flows y_{kt+1}^* of the hedging portfolios satisfy the
conditions

$$\mathbf{z}_t E_t(y_{kt+1} - y_{kt+1}^*) = \mathbf{0}_J \quad (49)$$

$$E_t[(\mathbf{r}_t \otimes \mathbf{z}_t)(y_{kt+1} - y_{kt+1}^*)] = \mathbf{0}_{NJ}, \quad (50)$$

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where $\mathbf{0}_J$ is a $J \times 1$ vector of zeros, whereas $\mathbf{0}_{NJ}$ is an $NJ \times 1$ vector of zeros. Assuming stationarity, and applying the law of iterated expectations, we have

$$E[\mathbf{z}_t(y_{kt+1} - y^*_{kt+1})] \;=\; \mathbf{0}_J \tag{51}$$

$$E\{[(\mathbf{r}_t \otimes \mathbf{z}_t)(y_{kt+1} - y^*_{kt+1})] \otimes \mathbf{z}_t\} \;=\; \mathbf{0}_{NJ}. \tag{52}$$

Hence, we have $y^*_{kt+1} \equiv \mathbf{z}_t^\top \gamma_{0k} + (\mathbf{r}_t^z)^\top \gamma_k$, where γ_{0k} and γ_k are a $J \times 1$ and an $NJ \times 1$ coefficient vector, respectively. The two coefficient vectors are estimated by method of moments imposing the conditions (49)-(50).[10] The excess cash flows of the hedging portfolios are given by

$$[(\mathbf{r}_{t+1} - r_{ft}\mathbf{1}_N) \otimes \mathbf{z}_t]^\top \gamma_k. \tag{53}$$

These cash flows are projected on the instruments $\mathbf{z}_t$ to obtain estimates of conditional and unconditional risk premia.

Finally, we estimate the Sharpe ratios of the hedging portfolios, since they might differ substantially from the conditional risk premia (see discussion above). Hence, we take the ratio between the risk premium and the standard deviation of the hedging-portfolios cash flows.

D. Multi-beta Models

We now turn to the issue of testing explicit asset-pricing models. Based on a variety of test we want to determine whether an MV kernel constructed from the cash-flows of the hedging portfolio, q_{y^*t+1}, prices all securities. The cash-flows of the hedging portfolios are given by $\mathbf{y}^*_{t+1} \equiv \gamma_0^\top \mathbf{z}_t + \gamma^\top \mathbf{r}^z_{t+1}$, where γ_0 and γ are a $J \times K$ and an $NJ \times K$ coefficient matrix, respectively. Each column of γ_{0y} (γ) is given by the coefficient vector γ_{0k} (γ_k). Now, note

---

[10]Closed-form expressions for the two coefficient vectors are given by:

$$\gamma_k \;=\; \{E[\mathsf{r}^z_{t+1}(\mathsf{r}^z_{t+1})^\top] - E(\mathsf{r}^z_{t+1}\mathsf{z}_t^\top)E(\mathsf{z}_t\mathsf{z}_t^\top)^{-1}E[\mathsf{z}_t(\mathsf{r}^z_{t+1})^\top]\}^{-1}\{E(\mathsf{r}^z_{t+1}\mathsf{y}^\top_{t+1}) - E(\mathsf{r}^z_{t+1}\mathsf{z}_t^\top)E(\mathsf{z}_t\mathsf{z}_t^\top)^{-1}E(\mathsf{z}_t)\}$$

$$\gamma_{0k} \;=\; E(\mathsf{z}_t\mathsf{z}_t^\top)^{-1}\{E(\mathsf{z}_t\mathsf{y}^\top_{t+1}) - E[\mathsf{z}_t(\mathsf{r}^z_{t+1})^\top]\gamma_k\}.$$

that the first component of the cash-flows, $\gamma_0^\top \mathbf{z}_t$, is certain, conditioning on $\mathbf{z}_t$. Hence, a kernel which conditionally has mean of one, prices this first component correctly.[11] As a result, the conditions that q_{y^*t} must satisfy are

$$E(\mathbf{z}_t q_{y^*t+1}) = E(\mathbf{z}_t) \tag{54}$$

$$\gamma^\top E(\mathbf{r}_{t+1}^z q_{y^*t+1}) = \gamma^\top E(\mathbf{r}_{t+1}^z - \mathbf{1}_t^z r_{ft}). \tag{55}$$

The resulting MV kernel has the form $q_{y^*t} = \mathbf{z}_t^\top \alpha_{0y} + (\gamma^\top \mathbf{r}_{t+1}^z)^\top \alpha_y$, where α_{0y} and α_y are a $J \times 1$ and a $K \times 1$ coefficient vector, respectively.

The χ^2 statistic is a test of the moment conditions

$$E(\mathbf{z}_t q_{y^*t+1}) = E(\mathbf{z}_t) \tag{56}$$

$$E(\mathbf{r}_{t+1}^z q_{y^*t+1}) = E(\mathbf{r}_{t+1}^z - \mathbf{1}_t^z r_{ft}). \tag{57}$$

We have $J + K$ coefficients in the vectors α_{0y} and α_y and $J + NJ$ moment conditions, for a total of $NJ - K$ overidentifying restrictions.

The HJV statistic is an estimate of the difference

$$\sqrt{\mathrm{Var}(q_{y^*t})} - \sqrt{\mathrm{Var}(q_t^*)}. \tag{58}$$

The HJD statistic is an estimate of the quantity

$$\sqrt{\mathrm{Var}(q_t^*) - \mathrm{Var}(q_{y^*t})}. \tag{59}$$

IV. Data

This section illustrates the data used in the empirical analysis. The period considered is March 1959-December 1996 for stock and bond returns and February 1959-November 1996 for economic and information variables.

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[11]In fact, $\mathrm{E}(\mathbf{z}_t q_{y^*t+1}) = \mathrm{E}(\mathbf{z}_t)$ implies $\mathrm{E}(\alpha_0^\top \mathbf{z}_t q_{c^*t+1}) = \mathrm{E}(\alpha_0^\top \mathbf{z}_t)$.

A. Asset Returns

We use decile portfolio returns on NYSE, AMEX, and NASDAQ listed stocks. Ten size stock portfolios are formed according to size deciles on the basis of the market value of equity outstanding at the end of the previous year. If a capitalization was not available for the previous year, the firm was ranked based on the capitalization on the date with the earliest available price in the current year. The returns are value-weighted averages of the firms's returns, adjusted for dividends. The securities with the smallest capitalizations are placed in portfolio one. The partitions on the CRSP file include all securities, excluding ADRs, that were active on NYSE-AMEX-NASDAQ for that year.

To be consistent with previous literature, we performed all our tests using twelve industry stock portfolios as well. The twelve industry stock portfolios are formed following the classification in Ferson and Harvey (1991)[12]. The industries are Petroleum, Finance/Real Estate, Consumer Durables, Basic Industries, Food/Tobacco, Construction, Capital Goods, Transportation, Utilities, Textile/Trade, Services, Leisure. The results for these tests are not reported in the tables, but they are discussed in the main text.

Together with the stock returns, we use bond portfolio returns. The bond portfolios are a long-term government bond, a long-term corporate bond, and the Treasury bill that is closest to 6 months to maturity. The long-term government and corporate bonds are provided by Ibbotson Associates, while the 6-month Treasury bill rate is from CRSP (Fama Treasury Bill Term Structure Files). The 1–month Treasury Bill rate chosen is from Ibbotson Associates SBBI module and pertains to a bill with at least 1 month to maturity.[13]

All rates of return are deflated using monthly inflation. The monthly rate of inflation is from SBBI Yearbook and is not seasonally adjusted.

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[12]We thank Campbell Harvey for kindly providing the FORTRAN codes necessary to form these twelve industry portfolios.

[13]In order to minimize measurement error problems, we use the 1-month bill rate from CRSP Fama Treasury Bill Term Structure Files. This methodology closely follows Ferson and Harvey (1991).

B. Economic Variables and Instruments

We concentrate on a set of seven variables which have been previously used in tests of multiple-beta models and/or in studies of stock-return predictability. (See, for example, Chen, Roll, and Ross (1986), Burmeister and McElroy (1988), Ferson and Harvey (1991, 1999), Downs and Snow (1994), and Kirby (1998)). These variables are statistically significant in multi-variate predictive regressions of means and volatilities and/or they have special economic significance:

INF is the monthly rate of inflation (Ibbotson Associates).

XEW represents the equally-weighted NYSE-AMEX-NASDAQ index return (CRSP) deflated by the monthly inflation rate from Ibbotson Associates.

CG denotes the logarithm of the monthly gross growth rate of per capita real consumption of nondurable goods and services. The series used to construct consumption data are from CITIBASE. Monthly real consumption of nondurable goods and services are the GMCN and GMCS series deflated by the corresponding deflator series GMDCN and GMDCS. Per capita quantities are obtained by using data on resident population, series POPRES.

HB3 is the 1-month return of a 3-month Treasury bill less the 1-month return of a 1-month bill (CRSP, Fama Treasury Bill Term Structure Files).

DIV denotes the monthly dividend yield on the Standard and Poor's 500 stock index (CITIBASE).

REALTB denotes the real 1–month Treasury bill (SBBI).

PREM represents the yield spread between Baa and Aaa rated bonds (Moody's Industrial from CITIBASE).

We select as instruments the lagged values of the previous seven variables as a proxy for the information investors use to set prices in the market. This choice of instruments mainly

follows Ferson and Harvey (1991). The reason for using as instruments the lagged values of the economic variables is that, according to the I-CAPM intuition, the variables that drive asset returns should also be the variables affecting the risk-return trade-off, i.e. they should also be the variables predicting returns.

V.   Results

Here we report results on: i) the analysis of predictability and heteroskedasticity; ii) the estimation of risk premia and Sharpe ratios; and iii) tests of multi-beta models.

A.   Predictability and Heteroskedasticity

Table I presents our results using decile and bond portfolio returns. We report the coefficients of the mean and variance equations, as well as three statistics. $\bar{\mu}_r$ is the average slope coefficient in the mean equations. $\bar{\sigma}_r$ is the average slope coefficient in the variance equations. $\bar{\mu}_r - \bar{\sigma}_r$ is the difference between the two average slope coefficients. These statistics provide an indication of the net effect of the instruments on the investment opportunity set.

The following patterns emerge from the analysis (see especially Panel C):

The inflation rate (INF) has a negative and significant average impact on returns; and a positive and significant average impact on return volatility. The net effect on the investment-opportunity set is strongly negative.

Lagged stock returns (XEW) have a positive and significant average impact on returns, and a negative and significant average impact on return volatility. The net effect is strongly positive: only the term structure variable HB3 has a stronger positive effect.

Consumption growth (CG) has an overall positive and significant effect on returns; and a negative, but insignificant, overall effect on volatility. The net effect is positive but very small.

The slope of the term structure (HB3) has an overall positive and significant impact on returns. The impact on volatility is negative and significant. The net effect is positive and large. Indeed, the slope of the term structure has the strongest positive effect on the investment-opportunity set among all instruments.

The dividend yield (DIV) affects positively and significantly returns. The overall effect on volatility is also positive and significant. The net effect is positive and fairly large.

The real rate of interest (REALTB) affects negatively and significantly returns, while the effect volatility is positive. The net effect is negative and large, although smaller than the inflation rate.

The default premium (PREM) has a positive impact on returns and a negative, but insignificant, impact on volatility. The net effect is positive, but smaller than that of all other variables, with the exception of consumption growth.

In summary, we can rank the net effects of the different variables on the investment-opportunity set as follows (from largest to smallest): HB3, XEW, DIV, PREM, CG, REALTB, INF. This preliminary analysis of predictability is useful because it allows us to establish a link between the effect of a variable on the risk-return trade-off, and the sign and size of the risk premium it commands. This link is new relative to existing studies and it may help us shed light on previous results.

The results from the same tests using industry-sorted stock portfolio returns are very similar. The only noteworthy difference is that all the predictability patterns tend to be more significant when we use size portfolios. This difference will appear in all tests, a possible indication that returns tend to behave more alike within a capitalization sector than within an industry.

B.   Risk Premia and Sharpe Ratios

Table II reports estimates of the coefficients of the economic risk premia estimated using the minimum-variance kernel q_t^*. Notice that since the instruments are demeaned, the intercept

term can be interpreted as the unconditional risk premium on y_{kt}.

Table III reports results for the risk premia assigned by the non-negative kernel $\tilde{q}_t$.

Table IV reports coefficient estimates of the economic risk premia estimated using the hedging portfolios.

The premia assigned by q_t^* and $\tilde{q}_t^*$ differ because the second kernel is more volatile than the first one. The additional volatility of $\tilde{q}_t$ may generate a correlation with the component of y_{kt} which is orthogonal to asset returns, and hence generate a discrepancy between λ_t^* and $\tilde{\lambda}_t$.

The expected excess cash-flows on the hedging portfolios are identical to the risk premia assigned by q_t^*. Yet, the realized excess cash flows on the hedging portfolios in general differ from $(q_t^* - 1)y_{kt}$. Hence, the estimates of the unconditional risk premia using the "q^*" and the "hedging-portfolio" approaches will coincide, although their standard errors may differ. In addition, the impact of the conditioning variables on the conditional risk premia will also differ.

The tables report two sets of T-ratios. The first T-ratio is obtained using a two-step procedure: we first estimate the coefficients of q_t^*, $\tilde{q}_t$, and y_{kt}^*. We then estimate the risk premia by GMM. The second t-ratio is obtained estimating all parameters by GMM. The reason for the two separate approaches is that we are concerned with the large number of estimated parameters when the coefficients of the minimum-variance kernels and the hedging portfolios are estimated by GMM. As it turns out, the T-ratios change only marginally across the two procedures. In all tests, standard errors are adjusted for heteroskedasticity of unknown form and serial correlation (MA of order 11).

The results are fairly similar across the three approaches. The signs of the unconditional premia are the same, while the absolute size of the unconditional premia, as well as their significance, tend to be slightly higher when using the $\tilde{q}_t$ approach. The patterns of time variation, on the other hand, tend to be stronger for the risk premia assigned by q_t^* and for the hedging-portfolios excess cash-flows.

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In the following discussion we focus on the results from the hedging-portfolio approach, Table IV. We do this for two reasons. First, standard errors tend to be tight, and several coefficients are significant. Second, these premia correspond to actual average excess cash flows, which are also one of the inputs to the Sharpe ratios estimated in the next exercise.

The following patterns emerge from the tables:[14]

The unconditional inflation premium is negative, as one would expect given the negative net impact of inflation on the investment-opportunity set. The premium equals -11 basis points using the q^* and hedging portfolio approaches; it equals -13 basis points using the $\tilde{q}$ approach.

There is also evidence of significant time-variation. The inflation premium is less negative for a higher real rate of interest: an increase in REALTB by one standard deviation increases the premium by 35 to 55 basis points. The premium is more negative for a steeper term structure: a one-standard deviation increase leads to an fall in the premium between.

The unconditional market risk premium is positive and significant, consistent with its positive net effect on investment opportunities.

The premium increases with stock returns, the slope of the term structure, and the dividend yield; the premium decreases with inflation and the real rate.

The unconditional consumption risk premium is positive, but insignificant. This result is consistent with the somewhat weak positive effect of consumption growth on investment opportunities.

Interestingly, although the unconditional consumption risk premium is insignificant, there is significant time-variation. Namely, inflation and the real rate affect negatively the conditional consumption risk premium.

The unconditional risk premium on the slope of the term structure is positive and

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[14]The discussion is based on the T-ratios obtained estimating the portfolio weights inside the GMM algorithm.

significant.[15] This is consistent with the evidence that a steeper yield curve has a positive effect on investment opportunities.

The premium increases with the inflation rate and decreases with stock returns.

The dividend-yield unconditional premium is negative and insignificant. This evidence can be reconciled with the relatively small net effect of this variable on investment opportunities.

As with the consumption premium, there is significant time-variation. The premium becomes less negative as inflation and the real rate increase. The premium becomes more negative as stock returns, the slope of the term structure, and the dividend yield increase.

The real rate of interest commands a positive unconditional risk premium. This result may appear puzzling, given the negative net effect on the risk-return trade-off. Yet, investors should care about both the slope and the position of the capital allocation line (CAL). A higher real rate of interest means an upward shift of the CAL, which may more than compensate the negative effect on the slope of the CAL.

As with other risk premia, there is significant time variation. The premium increases with the slope of the term structure and decreases with the real rate.

Finally, the default premium receives an insignificant positive unconditional risk premium. This is consistent with the mildly positive net effect on investment opportunities.

The premium increases with the dividend yield.

The results from the tests using industry-sorted portfolios are very similar.

Based on the results above, we can draw a few conclusions:

First, the sign and significance of the unconditional risk premia associated with the selected economic variables are largely consistent with the predictability patterns previously

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[15]It is positive, but insignificant, when estimated using the $\tilde{q}_t$ approach.

documented. Hence, there is support for Merton's (1973) I-CAPM intuition.

Second, conditional risk premia exhibit significant time variation. The time-varying patterns are estimated more precisely than in other studies. For example, Ferson and Harvey (1991) also document time variation in the market premium. They only find two significant effects, though: the premium increases with the dividend yield and decreases with the real rate. Our results show that two other variables affect significantly the market premium: the slope of the term structure (positive effect) and the rate of inflation (negative effect).

The risk premia estimated above coincide with the Sharpe ratios of exact hedging portfolios. But, in general, economic risk variables can be tracked only imperfectly by asset returns. Hence, in order to obtain Sharpe ratios on traded portfolios we need to standardize the estimates obtained above by the volatility of the approximate hedging portfolio cash flows. The composition of the hedging portfolios is estimated separately from the Sharpe ratios.

Table V presents unconditional risk premia on the seven hedging portfolios, the unconditional volatility on the portfolio cash-flows, and unconditional Sharpe ratios. In interpreting the results, it is useful to recall that the volatility of the portfolio cash-flows equals the square-root of the R^2 coefficient of a regression of the economic variables on the asset returns and a constant. Hence, the Sharpe ratios are simply the products of the risk premia and the reciprocals of the square roots of the R^2 coefficients.

The Sharpe ratios allow us to effectively rank the importance of the seven risk variables:

Interestingly, the largest Sharpe ratio is not associated with the market portfolio proxy, XEW, but with the term-structure variable HB3: .19 (.23 using industry-portfolio returns). The Sharpe ratio is about 20% higher than the risk premium.

The second largest Sharpe ratio is associated with the real rate of interest, REALTB, .16, which is about 30% higher than the corresponding risk premium.

Then we have the market proxy, with .12, and consumption growth, with .09. The market proxy is tracked almost exactly by its hedging portfolio,and hence the premium

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and the Sharpe ratio essentially coincide. The consumption-growth premium, on the other hand, is 91% higher than the corresponding premium.

The Sharpe ratio on the dividend yield is positive, small in absolute value, and insignificant.

The smallest Sharpe ratio, but the second largest in absolute value, is associated with the inflation rate, $-.16$ (52% higher, in absolute value, than the corresponding premium).

We can also compare the ranking of the economic variables based on their Sharpe ratios to that based on their net effect on investment opportunities. In both rankings, HB3 is the first variable, INF is the last one, and CG ranks fifth. The rankings of the other variables, on the other hand, are somewhat different.

The results from the industry-portfolio returns are roughly similar, although somewhat less significant.

C.   Multi-beta Models

In this section we discuss the results of tests of the four multi-beta models: C-CAPM, S-CAPM, I-CAPM, and FF.[16] Results of the tests are presented in Table VI. The tests are performed using the full set of instruments $\mathbf{z}_t$ ("With conditional information") and using only the constant z_1 ("Without conditional information") to scale asset returns.

We report three statistics: i) the χ^2 statistic associated with a test of the overidentifying restrictions; ii) the difference between the standard deviation of the candidate pricing kernel and the standard deviation of q_t^*, the HJV statistic; and iii) the Hansen-Jagannathan (1997) distance measure, the HJD statistic. We also report the p-values associated with the χ^2 test, and the T-ratios associated with the HJV and HJD statistics.

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[16] We obtained from Fama his monthly series of the size and book-to-market factors for the 1963-1993 period. We construct mimicking portfolios for the two factors using the 1963-1993 sample, and we performed the tests using the mimicking-portfolio cash flows for the full 1959-1996 period.

In the test of overidentifying restrictions, the coefficients of the candidate kernel are estimated by GMM, although the composition of the mimicking portfolios is estimated separately, outside of the GMM algorithm. This is because we wanted to ensure that the coefficients of the mimicking portfolios exactly satisfied the orthogonality conditions (51)-(52). The shortcoming of this approach is that the size of the test is not adjusted for the sampling variability in the estimates of the hedging-portfolio weights, and hence the test tends to reject too often.

In the tests based on the HJV and HJD, the coefficients of the candidate kernel are estimated separately from the statistics themselves. This means that the test take the form of the candidate kernel as given.

Overall, the table shows that while all four models are formally rejected, in both conditional and unconditional tests, their performance varies considerably. Across tests, the I-CAPM is the best performer, followed by the FF model, the S-CAPM, and the C-CAPM. In most tests, though, the difference in performance between the C-CAPM and the S-CAPM is modest. Comparing conditional and unconditional tests, the conditional tests tend to de-emphasize the differences across models and to lead to stronger rejections that the unconditional tests.

We now turn to a discussion of the different tests. We begin with the conditional tests.

The χ^2 tests strongly reject all four models. Interestingly, the χ^2 statistics are very similar for the C-CAPM and the S-CAPM, 191.91 and 192.88, respectively. The FF model has only a slightly lower χ^2 statistic, 187.39, whereas the I-CAPM has the lowest statistic, 168.42, although the rejection is still very strong. Hence, the performance of the four models is remarkably similar. One way to understand this feature is to note the total number of scaled returns that the restricted MV kernel must price is very large relative to the number of factors: we have $N = 13$ and $J = 8$, for a total of 104 scaled returns, while the number of factors ranges from 1 (C-CAPM and S-CAPM) to 5 (FF) and 7 (I-CAPM). Hence, the addition of even 6 factors makes little difference when the total number of securities to price is so large.

The HJV tests also strongly reject all four models. For a better understanding of the economic magnitudes involved, it is worth noting that the standard deviation of the MV kernel constructed using the original scaled returns is 1.033. The standard deviations of the restricted MV kernels implied by the C-CAPM and S-CAPM are one order of magnitude smaller, 0.0959 and 0.1278, respectively. The restricted MV kernel implied by the FF model has a standard deviation of 0.2243, whereas the MV kernel of the I-CAPM kernel has a standard deviation of 0.3304. Hence, the comparison of standard deviations allows to better differentiate the four models. In particular, the increase in volatility going from the FF model to the I-CAPM amounts to more than 47%.

The HJD tests deliver essentially the same message as the comparison of standard deviations. While the highest HJD statistic is for the C-CAPM, 1.0274, the S-CAPM delivers a very similar result, 1.0239. The FF model leads to a statistic not very dissimilar, 1.0072. The lowest HJD statistics is for the I-CAPM, 0.9777.

We now turn to the unconditional tests.

The χ^2 statistic now favors the S-CAPM relative to the C-CAPM: 43.55 vs 51.39. The statistic for the FF model is substantially lower, 26.44. The I-CAPM performs best, with a statistic of 18.27. The p-values of the statistics are overall much higher than in the conditional tests, the highest being for the I-CAPM model, 0.56%. Hence, in unconditional tests, the χ^2 statistics better differentiate across models, with the I-CAPM being markedly the best performer.

In interpreting the HJV tests it is worth noting that the standard deviation of the MV kernel constructed using the original returns is 0.3438. In comparison, the standard deviation of the MV kernel restricted by the C-CAPM is very low, 0.0179. Considerably higher is the standard deviation of the S-CAPM, 0.115. Further increases in volatility are obtained by the FF model, 0.2361, and especially by the I-CAPM, 0.2789. Hence, we have again a marked variation in performance across models, with the I-CAPM being by far the bet performer.

Finally, we examine the results of HJD tests. The statistics for the C-CAPM and S-CAPM are fairly similar, 0.3429 and 0.3234, respectively. Substantially lower are the statis-

tics for the FF model, 0.2493, and for the C-CAPM model, 0.2008.

While not reported in the table, we also performed the tests using industry-sorted stock portfolios. As in the other tests, using industry-sorted portfolio returns introduces more noise in the estimation. This translates into less precise estimates and, in this case, in somewhat less dramatic rejections of the multi-beta models.

VI.   Conclusions

This paper presents a new approach for the estimation of risk premia associated with observable sources of risk, which is based on the moments of the minimum-variance kernel of Hansen and Jagannathan (1991). Consistent with the I-CAPM intuition, we find that variables that significantly affect the position of the investment-opportunity set (the conditional means and volatilities of asset returns) also tend to receive non-zero risk premia. Moreover, variables that positively (negatively) affect the Sharpe ratio tend to receive positive (negative) risk premia.

We also provide extensive evidence on the performance of explicit asset-pricing models: the C-CAPM, the S-CAPM, the I-CAPM, and the FF model. While all models are formally rejected, the FF model and the I-CAPM perform substantially better than the static CAPM and the consumption CAPM. In addition, we find that the I-CAPM consistently outperforms the FF model.

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Table I

Predictability and Heteroskedasticity

We report estimates of a predictive model of the conditional mean and volatility of the returns on ten size-sorted equity portfolios, $r_1 \ldots r_{10}$, and on three bond portfolios, r_{TB6}, r_{CORP}, and r_{GOV}. Both conditional mean and conditional volatility are assumed to be linear linear functions of the lagged economic variables. INF denotes the monthly rate of inflation (percentage points per month). XEW is the equally-weighted NYSE-AMEX-NASDAQ index return less the monthly inflation rate (percentage points per month). CG is the monthly growth rate of per-capita real consumption of nondurables and services (percentage points per month). HB3 is the 1-month return of a 3-month Treasury bill less the 1-month T-bill rate (percentage points per month). DIV is the monthly dividend yield on the Standard and Poor's 500 stock index (percentage points per month). REALTB is the real 1-month Treasury bill rate (percentage points per month). PREM represents the yield spread between Baa and Aaa rated bonds (percentage points per month). T-statistics, in parentheses, are adjusted for heteroskedasticity and serial correlation. The sample period is 1959:3-1996:11.

Panel A: Slope Estimates of Mean Equations

Var	INF	XEW	CG	HB3	DIV	REALTB	PREM
r_1	−2.148 (−3.219)	2.328 (6.111)	0.074 (0.992)	1.057 (2.164)	1.277 (2.324)	−1.104 (−1.959)	0.251 (0.601)
r_2	−2.491 (−3.918)	1.905 (5.867)	−0.005 (−0.077)	1.026 (2.144)	1.448 (2.935)	−1.526 (−2.892)	0.187 (0.486)
r_3	−2.454 (−3.894)	1.560 (4.902)	0.024 (0.418)	1.095 (2.272)	1.437 (2.968)	−1.501 (−2.888)	0.254 (0.674)
r_4	−2.584 (−4.105)	1.220 (3.952)	0.064 (1.085)	1.121 (2.329)	1.448 (3.006)	−1.610 (−3.086)	0.297 (0.797)
r_5	−2.565 (−4.235)	1.024 (3.471)	0.029 (0.402)	1.150 (2.379)	1.440 (3.031)	−1.566 (−3.125)	0.331 (0.915)
r_6	−2.575 (−4.255)	0.855 (3.134)	0.048 (0.580)	1.231 (2.685)	1.449 (3.136)	−1.646 (−3.284)	0.395 (1.139)
r_7	−2.426 (−4.162)	0.694 (2.660)	0.024 (0.328)	1.263 (2.729)	1.302 (2.921)	−1.494 (−3.088)	0.421 (1.239)
r_8	−2.386 (−4.383)	0.530 (2.141)	0.040 (0.426)	1.302 (2.940)	1.191 (2.847)	−1.574 (−3.431)	0.578 (1.801)
r_9	−2.258 (−4.313)	0.324 (1.407)	0.057 (0.822)	1.297 (3.015)	0.994 (2.497)	−1.505 (−3.448)	0.627 (2.052)
r_{10}	−1.627 (−3.717)	0.068 (0.302)	0.005 (0.146)	0.987 (2.720)	0.532 (1.588)	−0.958 (−2.495)	0.519 (1.867)
r_{TB6}	0.020 (0.460)	−0.064 (−3.725)	−0.004 (−0.272)	0.028 (1.057)	−0.008 (−0.316)	0.146 (3.960)	0.045 (1.797)
r_{CORP}	−0.823 (−2.964)	−0.308 (−2.528)	−0.066 (−1.359)	1.085 (4.028)	0.161 (0.940)	−0.329 (−1.267)	0.231 (1.341)
r_{GOV}	−0.666 (−2.233)	−0.282 (−2.062)	−0.096 (−2.530)	1.171 (3.938)	0.090 (0.513)	−0.231 (−0.813)	0.163 (0.864)

Panel B: Slope Estimates of Volatility Equations

Var	INF	XEW	CG	HB3	DIV	REALTB	PREM
r_1	0.259 (0.590)	0.127 (0.487)	0.026 (0.159)	−0.114 (−0.325)	0.362 (0.884)	0.200 (0.531)	−0.243 (−0.880)
r_2	0.569 (1.384)	−0.071 (−0.331)	0.021 (0.144)	−0.339 (−1.061)	0.548 (1.603)	0.310 (0.893)	−0.235 (−0.958)
r_3	0.475 (1.205)	−0.217 (−1.025)	−0.001 (−0.009)	−0.337 (−1.067)	0.525 (1.606)	0.141 (0.431)	−0.183 (−0.774)
r_4	0.486 (1.272)	−0.255 (−1.246)	−0.014 (−0.091)	−0.322 (−1.023)	0.553 (1.719)	0.097 (0.305)	−0.141 (−0.615)
r_5	0.566 (1.498)	−0.394 (−2.062)	−0.030 (−0.235)	−0.425 (−1.377)	0.467 (1.516)	0.152 (0.487)	−0.137 (−0.627)
r_6	0.564 (1.502)	−0.529 (−2.957)	−0.019 (−0.165)	−0.355 (−1.225)	0.319 (1.081)	0.077 (0.250)	−0.114 (−0.545)
r_7	0.574 (1.586)	−0.676 (−4.127)	−0.018 (−0.147)	−0.444 (−1.543)	0.336 (1.206)	0.079 (0.265)	−0.043 (−0.214)
r_8	0.522 (1.574)	−0.643 (−4.204)	−0.007 (−0.082)	−0.391 (−1.417)	0.252 (0.975)	0.069 (0.250)	0.093 (0.506)
r_9	0.425 (1.304)	−0.766 (−5.405)	0.000 (0.001)	−0.367 (−1.345)	0.267 (1.113)	−0.019 (−0.070)	0.126 (0.705)
r_{10}	0.058 (0.218)	−0.732 (−5.437)	0.007 (0.055)	−0.233 (−0.963)	0.348 (1.692)	−0.124 (−0.526)	0.192 (1.118)
r_{TB6}	0.114 (4.276)	0.012 (1.141)	−0.005 (−0.880)	0.022 (1.574)	0.009 (0.568)	0.069 (3.195)	0.008 (0.539)
r_{CORP}	0.815 (4.651)	−0.042 (−0.521)	−0.023 (−1.507)	−0.190 (−0.985)	−0.086 (−0.758)	0.662 (4.112)	0.176 (1.569)
r_{GOV}	0.940 (4.987)	−0.031 (−0.341)	−0.021 (−0.588)	−0.108 (−0.483)	−0.339 (−3.070)	0.853 (4.793)	0.311 (2.531)

Panel C: Average Slope Estimates and Differences in Average Slope Estimates

Var	INF	XEW	CG	HB3	DIV	REALTB	PREM
$\bar{\mu}_r$	−1.922 (−7.290)	0.756 (2.876)	0.015 (0.057)	1.063 (4.030)	0.982 (3.723)	−1.146 (−4.347)	0.331 (1.255)
$\bar{v}_r$	0.490 (3.146)	−0.324 (−2.083)	−0.006 (−0.041)	−0.278 (−1.780)	0.274 (1.760)	0.198 (1.269)	−0.015 (−0.094)
$\bar{\mu}_r - \bar{v}_r$	−2.412 (−4.528)	1.080 (4.682)	0.021 (0.172)	1.340 (2.635)	0.708 (1.879)	−1.344 (−2.949)	0.346 (1.174)

Table II
Economic Risk Premia: q^* Approach

We report coefficients of the economic risk premia on the economic variables. INF denotes the monthly rate of inflation (percentage points per month). XEW is the equally-weighted NYSE-AMEX-NASDAQ index return less the monthly inflation rate (percentage points per month). CG is the monthly growth rate of per-capita real consumption of nondurables and services (percentage points per month). HB3 is the 1-month return of a 3-month Treasury bill less the 1-month T-bill rate (percentage points per month). DIV is the monthly dividend yield on the Standard and Poor's 500 stock index (percentage points per month). REALTB is the real 1-month Treasury bill rate (percentage points per month). PREM represents the yield spread between Baa and Aaa rated bonds (percentage points per month). T-statistics, in parentheses, are adjusted for heteroskedasticity and serial correlation. The t-statistics refer to the case where the composition of q^* is estimated outside and inside the GMM algorithm, respectively. The sample period is 1959:3-1996:11. For ease of comparison, we also report the differences in the average slope coefficients of the mean and volatility equations from Table I, Panel C.

Risk Premia	λ_{INF}	λ_{XEW}	λ_{CG}	λ_{HB3}	λ_{DIV}	λ_{REALTB}	λ_{PREM}
Intercept	−0.1079 (−1.83) (−2.80)	0.1179 (2.15) (2.70)	0.0492 (1.43) (1.17)	0.1612 (2.30) (3.22)	−0.0086 (−0.15) (−0.72)	0.1200 (2.32) (2.73)	0.0282 (0.47) (1.42)
z_{INF}	0.4954 (2.74) (3.70)	−0.3980 (−2.59) (−3.88)	−0.1423 (−2.85) (−2.97)	0.4868 (2.04) (3.18)	0.1169 (0.65) (0.86)	−0.2589 (−1.92) (−2.00)	−0.0433 (−0.21) (−0.27)
z_{XEW}	0.0182 (0.28) (0.33)	0.1980 (3.18) (4.19)	0.0100 (0.56) (0.54)	−0.1814 (−2.03) (−3.33)	−0.0637 (−1.04) (−1.25)	−0.0971 (−1.67) (−1.98)	−0.0559 (−0.82) (−1.08)
z_{CG}	0.0095 (0.63) (0.79)	0.0081 (0.80) (0.75)	−0.5415 (−1.44) (−3.40)	0.0038 (0.11) (0.20)	0.0074 (0.22) (0.42)	−0.0115 (−0.22) (−0.37)	0.0197 (1.04) (1.87)
z_{HB3}	−0.0688 (−0.57) (−0.98)	0.2061 (2.53) (2.57)	−0.0120 (−0.73) (−1.19)	0.1503 (0.73) (1.32)	0.1021 (0.84) (1.66)	0.1128 (1.40) (1.46)	0.2920 (2.56) (3.87)
z_{DIV}	−0.0045 (−0.05) (−0.06)	0.2230 (2.18) (2.80)	−0.0010 (−0.05) (−0.05)	−0.0446 (−0.44) (−0.56)	0.0393 (0.33) (0.45)	−0.0419 (−0.48) (−0.58)	0.0008 (0.00) (0.00)
z_{REALTB}	0.5495 (3.82) (4.67)	−0.2712 (−2.38) (−3.18)	−0.1196 (−2.46) (−2.19)	0.1170 (0.86) (1.00)	−0.0036 (−0.02) (−0.03)	−0.5175 (−3.88) (−4.45)	−0.1344 (−0.78) (−0.99)
z_{PREM}	−0.1922 (−1.71) (−2.52)	0.0543 (0.51) (0.90)	0.0229 (1.18) (1.20)	−0.1036 (−0.82) (−1.03)	−0.0161 (−0.13) (−0.18)	0.1930 (1.97) (2.86)	0.0874 (0.54) (1.08)

Var	INF	XEW	CG	HB3	DIV	REALTB	PREM
$\bar{\mu}_r - \bar{v}_r$	−2.412 (−4.528)	1.080 (4.682)	0.021 (0.172)	1.340 (2.635)	0.708 (1.879)	−1.344 (−2.949)	0.346 (1.174)

Table III

Economic Risk Premia: $\tilde{q}$ Approach

We report coefficients of the economic risk premia on the economic variables. INF denotes the monthly rate of inflation (percentage points per month). XEW is the equally-weighted NYSE-AMEX-NASDAQ index return less the monthly inflation rate (percentage points per month). CG is the monthly growth rate of per-capita real consumption of nondurables and services (percentage points per month). HB3 is the 1-month return of a 3-month Treasury bill less the 1-month T-bill rate (percentage points per month). DIV is the monthly dividend yield on the Standard and Poor's 500 stock index (percentage points per month). REALTB is the real 1-month Treasury bill rate (percentage points per month). PREM represents the yield spread between Baa and Aaa rated bonds (percentage points per month). T-statistics, in parentheses, are adjusted for heteroskedasticity and serial correlation. The t-statistics refer to the case where the composition of $\tilde{q}$ is estimated outside and inside the GMM algorithm, respectively. The sample period is 1959:3-1996:11. For ease of comparison, we also report the differences in the average slope coefficients of the mean and volatility equations from Table I, Panel C.

Risk Premia	λ_{INF}	λ_{XEW}	λ_{CG}	λ_{HB3}	λ_{DIV}	λ_{REALTB}	λ_{PREM}
Intercept	−0.1306 (−1.95) (−3.31)	0.1220 (1.94) (2.80)	0.1004 (1.29) (1.24)	0.1018 (1.45) (1.40)	−0.0118 (−0.18) (−1.05)	0.1550 (2.24) (3.32)	0.0386 (0.62) (2.01)
z_{INF}	0.1545 (0.95) (1.07)	−0.4151 (−3.01) (−4.02)	−0.1501 (−2.07) (−2.04)	0.1699 (1.00) (1.12)	−0.0382 (−0.25) (−0.26)	−0.1063 (−0.65) (−0.88)	0.0046 (0.032) (0.035)
z_{XEW}	0.0306 (0.36) (0.50)	0.2043 (2.70) (4.31)	0.0328 (1.38) (1.54)	−0.1911 (−2.38) (−3.37)	−0.0014 (−0.02) (−0.02)	−0.0603 (−0.73) (−0.97)	−0.0943 (−1.52) (−1.87)
z_{CG}	0.0105 (0.77) (1.16)	0.0080 (0.80) (0.80)	−0.4869 (−1.46) (−1.47)	0.0032 (0.10) (0.21)	0.0095 (0.29) (0.46)	−0.0136 (−0.27) (−0.48)	0.0129 (0.65) (1.28)
z_{HB3}	−0.0813 (−0.98) (−1.24)	0.2092 (3.00) (2.62)	−0.0035 (−0.28) (−0.30)	0.0565 (0.53) (0.58)	0.1385 (1.54) (1.92)	0.1505 (1.86) (2.24)	0.2228 (2.75) (2.95)
z_{DIV}	0.0382 (0.35) (0.50)	0.2219 (2.30) (2.80)	−0.0431 (−1.00) (−1.00)	0.0702 (0.76) (0.67)	0.0341 (0.25) (0.31)	−0.0521 (−0.47) (−0.69)	−0.0502 (−0.56) (−0.65)
z_{REALTB}	0.3610 (2.53) (2.98)	−0.2810 (−2.75) (−3.28)	−0.1526 (−2.23) (−2.03)	−0.0378 (0.30) (−0.34)	−0.0862 (−0.60) (−0.63)	−0.4414 (−2.68) (−3.91)	−0.1042 (−0.76) (−0.79)
z_{PREM}	−0.0805 (−0.67) (−1.15)	0.0567 (0.75) (0.94)	0.0379 (1.38) (1.31)	−0.0194 (−0.20) (−0.22)	0.0200 (0.18) (0.23)	0.1521 (1.35) (2.17)	0.1257 (0.91) (1.34)

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Table IV

Economic Risk Premia: Mimicking Portfolios Approach

We report coefficients of the economic risk premia on the economic variables. INF denotes the monthly rate of inflation (percentage points per month). XEW is the equally-weighted NYSE-AMEX-NASDAQ index return less the monthly inflation rate (percentage points per month). CG is the monthly growth rate of per-capita real consumption of nondurables and services (percentage points per month). HB3 is the 1-month return of a 3-month Treasury bill less the 1-month T-bill rate (percentage points per month). DIV is the monthly dividend yield on the Standard and Poor's 500 stock index (percentage points per month). REALTB is the real 1-month Treasury bill rate (percentage points per month). PREM represents the yield spread between Baa and Aaa rated bonds (percentage points per month). T-statistics, in parentheses, are adjusted for heteroskedasticity and serial correlation. The t-statistics refer to the case where the composition of the mimicking portfolios is estimated outside and inside the GMM algorithm, respectively. The sample period is 1959:3-1996:11.

Risk Premia	λ_{INF}	λ_{XEW}	λ_{CG}	λ_{HB3}	λ_{DIV}	λ_{REALTB}	λ_{PREM}
Intercept	−0.1079 (−3.81) (−2.91)	0.1179 (2.71) (2.70)	0.0492 (2.04) (1.52)	0.1612 (4.70) (3.28)	−0.0086 (−1.63) (−1.27)	0.1200 (3.63) (2.74)	0.0282 (2.93) (1.88)
z_{INF}	0.1535 (2.21) (1.85)	−0.4123 (−3.99) (−4.01)	−0.1237 (−3.14) (−2.23)	0.2907 (2.63) (2.32)	0.0517 (3.52) (3.12)	−0.1645 (−2.02) (−1.71)	−0.0402 (−1.27) (−1.08)
z_{XEW}	0.0389 (1.16) (1.01)	0.1926 (4.04) (4.05)	0.0307 (2.12) (0.91)	−0.2067 (−5.25) (−4.14)	−0.0192 (−3.12) (−2.72)	−0.0588 (−1.46) (−1.25)	−0.0183 (−1.32) (−1.08)
z_{CG}	0.0107 (1.19) (0.09)	−0.0200 (−2.02) (−0.89)	0.0305 (0.09) (0.08)	0.0240 (1.04) (0.15)	−0.0130 (−5.42) (−0.54)	−0.0317 (−2.06) (−0.24)	−0.0485 (−4.75) (−0.85)
z_{HB3}	−0.1324 (−3.48) (−2.56)	0.2013 (2.50) (2.48)	0.0169 (2.22) (0.69)	−0.0415 (−0.47) (−0.41)	−0.0331 (−3.48) (−2.94)	0.1332 (3.05) (2.17)	0.0535 (1.82) (1.49)
z_{DIV}	−0.0476 (−1.07) (−0.91)	0.2208 (2.76) (2.74)	−0.0062 (−0.47) (−0.34)	−0.0851 (−1.57) (−1.39)	−0.0282 (−2.59) (−2.28)	0.0318 (0.60) (0.52)	0.0519 (3.06) (2.51)
z_{REALTB}	0.3498 (5.20) (4.52)	−0.2853 (−3.35) (−3.35)	−0.1321 (−3.28) (−2.40)	−0.0478 (−0.58) (−0.49)	0.0402 (3.24) (2.85)	−0.4174 (−5.54) (−4.76)	−0.0043 (−0.13) (−0.11)
z_{PREM}	−0.0825 (−2.00) (−1.76)	0.0656 (1.09) (1.08)	0.0254 (1.83) (1.06)	0.0384 (0.64) (0.55)	−0.0077 (−0.79) (−0.73)	0.0918 (1.90) (1.69)	−0.0291 (−1.63) (−1.31)

Table V

Unconditional Risk Premia, Volatility of Hedging-Portfolio Cash Flows, and Sharpe Ratios

We report unconditional economic risk premia (λ_0), volatilities of the mimicking portfolios' excess cash flows (v_0) and Sharpe ratios ($S_{y^*_k}$) commanded by the economic variables. INF denotes the monthly rate of inflation (percentage points per month). XEW is the equally-weighted NYSE-AMEX-NASDAQ index return less the monthly inflation rate (percentage points per month). CG is the monthly growth rate of per-capita real consumption of nondurables and services (percentage points per month). HB3 is the 1-month return of a 3-month Treasury bill less the 1-month T-bill rate (percentage points per month). DIV is the monthly dividend yield on the Standard and Poor's 500 stock index (percentage points per month). REALTB is the real 1-month Treasury bill rate (percentage points per month). PREM represents the yield spread between Baa and Aaa rated bonds (percentage points per month). T-statistics, in parentheses, are adjusted for heteroskedasticity and serial correlation. The t-statistics refer to the case where the composition of y^* is estimated outside the GMM algorithm. The sample period is 1959:3-1996:11.

Variables	Decile and Bond Port.		
Sharpe Ratios	λ_0	v_0	$S_{y^*_k}$
INF	−0.1079 (−3.51)	0.6548 (21.84)	−0.1648 (−3.32)
XEW	0.1179 (2.53)	0.9910 (18.05)	0.1189 (2.49)
CG	0.0492 (2.00)	0.5235 (3.19)	0.0940 (4.31)
HB3	0.1612 (4.15)	0.8271 (11.10)	0.1949 (5.06)
DIV	−0.0086 (−1.50)	−0.1225 (−17.06)	0.0707 (1.49)
REALTB	0.1200 (3.32)	0.7683 (23.07)	0.1562 (3.18)
PREM	0.0282 (2.68)	0.2235 (9.75)	0.1261 (3.02)

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Table VI

Tests of C-CAPM, S-CAPM, I-CAPM and FF

In the first three columns of the tables we perform conditional tests of the Consumption Capital Asset Pricing Model (C-CAPM), the Static Capital Asset Pricing Model (S-CAPM), the Intertemporal Capital Asset Pricing Model (I-CAPM) and the Fama-French (FF) five-factor model, by using region subset tests (χ^2), Hansen-Jagannathan variance bounds (HJV), and Hansen-Jagannathan distance measures (HJD). The same statistics are reported in the last three columns, where the tests are unconditional. The standard deviation of the MV kernel is 1.0330, conditioning information, and 0.3438, no conditioning information. The standard deviations of the restricted MV kernels are reported in columns two and five, respectively.

Models	With conditional information			Without conditional information		
Statistics	$\chi^2_{(dof)}$ $(p-value)$	HJV $(t-stat.)$ $std(q_{y*})$	HJD $(t-stat.)$	$\chi^2_{(dof)}$ $(p-value)$	HJV $(t-stat.)$ $std(q_{y*})$	HJD $(t-stat.)$
C-CAPM	$191.91_{(103)}$ $(2.51e^{-07})$	-0.9361 (-19.12) $std(q_{y*})=0.0959$	1.0274 (26.47)	$51.39_{(12)}$ $(7.92e^{-07})$	-0.3254 (-22.53) $std(q_{y*})=0.0179$	0.3429 (23.79)
S-CAPM	$192.88_{(103)}$ $(1.98e^{-07})$	-0.9043 (-23.43) $std(q_{y*})=0.1278$	1.0239 (26.51)	$43.55_{(12)}$ $(1.81e^{-05})$	-0.2279 (-16.64) $std(q_{y*})=0.1155$	0.3234 (25.32)
I-CAPM	$168.42_{(97)}$ $(9.48e^{-06})$	-0.7019 (-18.33) $std(q^{y*})=0.3304$	0.9777 (28.31)	$18.27_{(6)}$ (0.0056)	-0.0648 (-6.12) $std(q_{y*})=0.2789$	0.2008 (22.76)
FF	$187.39_{(99)}$ $(2.02e^{-07})$	-0.8075 (-20.88) $std(q_{y*})=0.2243$	1.0072 (27.95)	$26.44_{(8)}$ (0.0009)	-0.1072 (-8.53) $std(q_{y*})=0.2361$	0.2493 (26.82)